Dynamic Certification and Reputation for Quality *

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Abstract

We study a firm investing in quality and building its reputation for quality. Quality can be certified at a cost. We consider two types of equilibria: one in which certification decisions are made based on firm's reputation and the second in which they are made based on the time since last certification. We show that reputation-based certification has a very limited effect on incentives to invest in quality, with investment only when the firm's reputation is the lowest. We also show that the firm in this case suffers from an over-certification trap in which the benefits of reputation are dissipated by excessive certification. These problems can be avoided with time-based certification, which can allow first-best investment in quality despite the investment being unobservable. We also show that the optimal certification duration results in the firm certifying when its reputation is high.

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1 Introduction

Customers in many industries rely on disclosure of product quality to make purchasing decisions. Such disclosure is often voluntary, and to be credible, is often provided or verified by a third party at a cost. Examples range from health care (for example, accreditation of HMOs by NCQA, described below), child care (for example, accreditation provided by the National Association for the Education of Young Children), and supplier relationships in B2B contracting (for example, ISO 9000 certification with over one million organizations independently certified worldwide).

In this paper we analyze how dynamic voluntary disclosure by firms affects their incentives to invest in quality and their resulting equilibrium reputation for quality. We show that time-contingent certifications provide better incentives for investments in quality than belief/reputation-contingent certifications, and that the former yield higher payoffs for the firms. We also show that certification can be a double-edged sword: on one hand it allows firms to reap benefits of investments in quality, on the other hand, it can create an (over) certification trap, if the market expects the firm to re-certify when its reputation drops sufficiently. Paradoxically, firms caught in such a trap earn lower profits than if no certification were possible, and they invest in quality sporadically, only after failing to certificate.

Firms can affect the quality of their products by investing in physical or human capital, research and development, or organizational design. Customers often do not directly observe these investments (or their results), giving rise to a moral hazard problem. That problem can be mitigated if the firm can build a reputation for quality, justifying premium prices or increased demand if the firm maintains its reputation. However, for the reputation to be credible, customers need to observe signals of quality and these are often provided by voluntary certification.[1] As in Board and Meyer-ter-Vehn (2013), we adopt a capital-theoretic approach to modeling both quality and reputation. The firm makes simultaneously investment and disclosure decisions. The firm can invest or disinvest in product quality, quality is persistent, and profit flow depends on firm's reputation, which is defined as market's belief about its quality.[2] This setting seems realistic for many markets. For example, in

[1]Other sources of information about product quality include mandatory disclosure (such as nutritional facts), third-party initiated reviews (such as reviews on Cnet.com), and consumer reports (word of mouth or consumer reports on Amazon.com). See a survey by Dranove and Jin (2010).

[2]Profits can increase in perceived quality either because good reputation leads to a bigger demand for the product or because it allows the firm to charge a higher price, or both. For empirical evidence that certification increases demand, see for example Xiao (2007) in the context of voluntary accreditation of child care centers, and other examples in Dranove and Jin (2010).

the healthcare industry, HMOs invest in processes and personnel to provide high-quality services, quality is persistent since human capital and organizational capital are persistent but maintaining quality requires continuous investment to attract and retain talent, and to react to changes in medical practice and technology. Moreover, quality is hard to observe by individual customers and a very important source of information is the National Committee of Quality Assurance (NCQA) that since 1991 offers HMOs voluntary certification program. The certificates expire in three years (that is, the decision to re-certify is time-based) and total costs (direct fees and indirect costs) of preparing accreditation range from $30,000$ to $100,000$ depending on the size of an HMO (and other characteristics; see Jin (2005) for a detailed description of the NCQA program).

In our model, firms continuously and privately choose quality investment. Quality changes stochastically between two states, high and low, with the transition rates depending on the instantaneous investment flows, so that current quality reflects all past investments. Investment affects quality in a way that if the firm is expected to be investing, its reputation for quality improves, and when it is expected to be not investing (or disinvesting), its reputation for quality is dropping. The firm sells its products continuously and its profit flow is proportional to its reputation. Quality is known privately by the firm but at any time it can be credibly revealed/certified to the market. We model certification as a costly disclosure (in the spirit of Jovanovic (1982), Verrecchia (1983)) that allows the firm to credibly and perfectly convey its current (and somewhat persistent) quality to the market. Though we do not model the source of this disclosure cost, we interpret it as representing the fee charged by a certifier in exchange for its certification and dissemination services (in the spirit of Lizzeri (1999)), plus any costs necessary to allow the certifier verify the firm's quality.

Since the firm is privately informed about its quality, the market learns about quality not only from certification but also from the failure to apply for accreditation. This leads to multiplicity of equilibria that vary in market beliefs about when the firm is expected to re-certify if its quality is still high. We look at two natural classes of equilibria: with belief-based policies and time-based policies. In the first class, the firm strategies (investment and certification) depend only on its current reputation. In the second class, they depend only on the time since the last re-certification. For the belief-based certification (the first class of equilibria), we show that, somewhat paradoxically, all equilibria suffer from the certification trap. There exist equilibria with positive investment in quality, but the firm invests in quality only to rebuild its reputation after failing to certificate, and never invests to maintain its high quality. Moreover, the firm is worse-off in the equilibrium with certification than it would be if no certification was possible. In fact, the equilibrium payoffs of the high-quality firm decrease (again, paradoxically) as the costs to certify go down; when certification becomes cheaper, the firm cannot stop itself from increasing the certification frequency. The intuition for these results is that in a belief-based certification, the firm re-certifies as soon as the difference in continuation payoffs between certifying and not is high enough to cover the certification costs. Moreover, the firm does not invest before its reputation drops below the certification threshold since otherwise beliefs would start increasing before certification and the threshold would never be reached.

For time-based certification, while the same problems still exist in some equilibria, there are also equilibria with less-frequent certification in which dynamic certification provides incentives for investment in maintaining quality and result in higher payoffs. While it may be at first counter-intuitive that less frequent certification improves incentives to invest in quality, the intuition is that with less frequent certification, the total expected continuation profits from certifying high quality are higher since less resources are spent on certifying. Moreover, there is a positive feedback effect: higher payoffs from high quality increase incentives for investment, which increases the payoffs even further.

The optimal (from the point of view of a high-quality firm) certification duration results, depending on parameter values, in either the firm always maintaining high quality (so that in equilibrium it always passes the certification), or the firm shirking on quality right after it certifies, but restarting investment before the time to recertify. Generally, the optimal duration of certification implies that the firm certifies when either its reputation is highest or when it is above its lowest point. It may seem at first as wasteful (especially in the first case), that the firm spends resources to certify even it is known that on the equilibrium path it always passes, but it is necessary to provide the right incentives for investing in quality.

In practice, firms can affect the market expectations about the frequency of certification (and hence try to select the good time-based equilibria) by self-regulation and/or employing the help of third parties to create certification with a pre-announced duration. Firms deviating and trying to re-certify prematurely can be either denied by the third-party certifier worried about creating a precedence in the industry, or punished by expectations that once they certify sooner than expected, the market expects them certify even more often in the future.[3]

[3] Such concerns for reputation for reticence are not revealing information too often are well known to managers in areas beyond certification. See for example Houston Lev and Tucker (2010) for voluntary earnings guidance by firms.

1.1 Related Literature

As we mentioned above, our paper can be viewed as a dynamic version of Jovanovic (1982),Verrecchia (1983) with endogenous quality. As in their models, we are interested in endogenous costly information disclosure. Unlike these models, the firm has the choice not only whether to certify, but also when. Furthermore, in contrast to these models, the firm chooses investment in quality. We show that in the best belief-based equilibria the high-quality firm is always hurt by the ability to certify in our dynamic model, something that is never true in a static model. This is a surprising result given the existing literature that has suggested that the distortions caused by the presence of asymmetric information may be solved by certifiers (see e.g., Biglaiser and Friedman (1994), Biglaiser (1993), Albano and Lizzeri (2001)). Moreover, we argue that if the duration of certificates is chosen optimally, then granting the firm access to certification dominates a no certification commitment, in contrast with settings without investment (see e.g., Jovanovic 1992, Verrecchia 1983, 1990). This result not only confirms the beneficial role of certification but is also consistent with the way certification is provided in practice. The bottom line is that certification can help a firm maintain reputation and make promises to invest in quality credible, but it can also lead to a certification trap.

Also as mentioned above, our model of evolution of quality and our interpretation of reputation as market's belief about firm's quality are the same as in Board and Meyer-ter Vehn (2013). [4] We differ from that paper in how we model information: in our model it is generated endogenously by the firm, while in their model the market observes exogenous signals about the quality. Moreover, while both papers ask about incentives to invest in quality, their paper asks how the nature of the imperfect exogenous signals affects incentives, while our paper asks how market expectations over when the firm will re-certify high quality affects incentives. On the more technical level, they study belief-contingent Markov-Perfect equilibria while we compare belief-contingent and time-contingent equilibria and show that in our setup the latter provide better incentives and payoffs.

A strand of the literature studies certification, focusing on the behavior of a monopoly certifier who can commit in advance to both a certification fee and a disclosure rule (see e.g., Lizzeri (1999), Albano and Lizzeri (2001)). In this paper we take the certification technology as exogenous and focus instead on firm's investment behavior, but we believe our model can be also used to study pricing decisions by a certifier. Our model suggests that as important

[4] See Mailath and Samuelson (2015) for a recent survey on the reputation literature.

decision as a price is the duration of certification and that the tradeoffs are non-trivial: longer duration can actually result in more certification if it provides stronger incentives to maintain quality.

Our paper is also somewhat related to the recent literature on reputation with information acquisition. (see e.g.,Liu (2011)), where it is the buyers who can acquire information about the firm. The main difference is that in our model quality is endogenous and persistent, and it is the firm that incurs costs to provide information.

Our model shares some features with the statistical discrimination literature initiated by Arrow (1973).[5] The underinvestment problem described in this paper is driven by the unobservability of quality and investment choices. The return to investment depends on the profits that the firm can assure by certifying high quality. In turn, these profits are determined by the buyers' expectation about future investments. In some sense, investment, certification, and buyers' beliefs are strategic complements, so that underinvestment becomes a self-fulfilling prophecy and longer duration of certification can lead to higher incentives to invest.

The remainder of the paper is organized as follows. In Section 2 we describe the model. In Section 3, we study equilibria when the firm chooses when to certify based on its current reputation. We contrast this case, in Section 4, with a setting where the certifiers choose duration of certification and the firm chooses when to certify based on the time since its last certification. We then discuss optimal duration of certification and the resulting patterns of investment and reputation.

2 Model

There is one firm and a competitive market of identical consumers. Time $t \in [0, \infty)$ is continuous. At every time t, the firm chooses privately investment in quality, makes decision about certification, and sells a product to the consumers, whose demand depends on perceived quality (firm's reputation).

We borrow the model of investment in quality from Board and Meyer-ter Vehn (2013). In particular, at time t, the firm's product quality is denoted by $\theta_t \in \{L, H\}$ where we normalize $L = 0$ and $H = 1$. Initial quality θ_0 is exogenous and commonly known, but subsequent quality depends on investment and unobservable technology shocks. Shocks are generated according to a Poisson process with arrival rate $\lambda > 0$. Quality θ_t is constant between shocks

[5]See Arrow (1998) for a review of the literature.

and is determined by the firm's investment at the most recent technology shock $s \leq t$ that is, $\theta_t = \theta_s$ and $\Pr(\theta_s = H) = a_s$. The firm observes product quality and chooses an investment plan $a = \{a_t\}_{t\geq 0}$, $a_t \in [0,1]$ which is predictable with respect to the filtration generated by $\theta = \{\theta_t\}_{t\geq 0}$. Investment has a marginal flow cost $k > 0$. Consumers observe neither quality nor investment. We denote their belief about the firm's investment by $\tilde{a} = \{\tilde{a}_t\}_{t\geq 0}$.

This specification implies that, given an investment policy a, quality jumps from L to H at an exponential time with arrival rate λa_t and jumps from H to L at a rate $\lambda(1 - a_t)$. As a consequence, investment has a persistent effect on product quality, as in the case when investment refers to employee training.[6]

Since λ measures the likelihood of shocks, a higher λ can be interpreted as capturing the instability of the firm's economic environment. On the technical side, note that since we assume $a_t \in [0,1]$, in the absence of investment the product quality can only experience negative shocks, and when investment is maximal, product quality can experience only positive shocks

To focus on the role of certification in reputation, unlike Board and Meyer-ter Vehn (2013), we assume that there are no public signals about firm quality. Instead, the firm has access to an external (unmodeled) party, referred to as the certifier, who can credibly certify the current quality of the firm for a fee c. Product quality becomes public information at the time of certification.

We denote the seller's certification strategy by $d_t \in \{0,1\}$ and the market beliefs by $\tilde{d}$. The firm is risk neutral and discounts future payoffs at rate $r > 0$. We model the market in a reduced form by assuming that the firm profit flow is a linear function of its reputation. Given the linearity assumption, without loss of generality, the profit flow at time t is $p_t = E^{\tilde{a},\tilde{d}}[\theta_t|\mathcal{F}_t^d]$ where $\mathcal{F}_t^d$ is the information generated by the firm's observed certification choices.

There are multiple ways to interpret this specification of profits. For example, as in Board and Meyer-ter Vehn (2013) the firm may be selling a limited amount of the product per period and the customers compete for the supply in a Bertrand fashion, which leads to prices being equal to the expected value of the product flow. Alternatively, the price may be fixed and the demand for the product may be proportional to the firm's reputation.

Given the firm's investment and certification strategy (a, d) and the market's belief about

[6]Also a retention and selection policy for employees has persistent effects on the quality of the workforce of a firm.

them $(\tilde{a}, \tilde{d})$, the firm's expected present value equals

$$E^{a,d,\theta_0}\left[\int_0^\infty e^{-rt}\big(p_t - a_t k\big)dt - \sum_{t_k \geq 0} e^{-rt_k}\, c \cdot d_{t_k}\right]$$

The conjectured investment and certification process $(\tilde{a}, \tilde{d})$ determines the firm's profit flow for a given history, while the actual strategy (a, d) determines the distribution over quality and histories.

Definition 1. *An equilibrium is a pair of strategies (a, d), and beliefs $(\tilde{a}, \tilde{d})$ such that given the market beliefs, the firm's strategy is optimal and beliefs are correct on the equilibrium path.*

Characterizing equilibria, throughout the paper we focus on pure strategy equilibria, in particular, in which the firm certification strategy, d, is pure.

Before studying the equilibrium, note that in the absence of disclosure, the evolution of beliefs is given by the ordinary differential equation

$$\dot{p}_t = \lambda\big(\tilde{a}_t - p_t\big).$$

When $\tilde{a}_t = 0$ the market beliefs p_t drift downward, and when $\tilde{a}_t = 1$, they drift upward. Throughout the paper, we assume that k is sufficiently small, $k < \frac{\lambda}{\lambda + r}$. This implies that $a_t = 1$ is the first best investment, namely the investment the firm would choose if either quality or investment were observable by the market.

There are several possible histories off-the-equilibrium path: the firm may certify sooner than expected, in which case we assume consumers believe the certification is truthful (so that beliefs have to re-set to $p_t = 1$). Moreover, the firm may fail to certify even if it is believed to have maintained high quality by investing $a_t = 1$. In that case, the beliefs are not restricted by the Bayes' rule.

In what follows, we study equilibria in two classes. First, in Section 3 we consider the belief-contingent equilibria (in which the investment and certification strategies depend on current beliefs) and then, in Section 4, we consider time-contingent equilibria (in which the strategies depend on the time since last certification).

3 Belief-Contingent Certification

In this section, we consider (pure) Belief-Contingent Markov-Perfect equilibria. That is, we study equilibria in which the firm strategy (a, d) is a function only of its current quality θ and the market belief p, and not of the full history of the game (in particular, it does not depend on the firm's actions before the last certification since we assume that every certification re-sets beliefs to $p_t = 1$ and recall that throughout the paper we restrict attention to pure certification strategies). Market expectations about firm certification and what investment strategies are hence a function only of the current belief p.

In state (θ, p) at the time when the firm is not expected to certify, its (continuation) value $V_\theta(p)$ satisfies the following HJB equations:

$$rV_L(p) = \max_{a\in[0,1]} p - ak + \lambda(\tilde{a}(p) - p)V_L'(p) + \lambda a D(p)$$

$$rV_H(p) = \max_{a\in[0,1]} p - ak + \lambda(\tilde{a}(p) - p)V_H'(p) - \lambda(1-a)D(p),$$

where we call $D(p) \equiv V_H(p) - V_L(p)$ the value of quality, namely the gain the seller experiences when its quality improves, given the market belief is p. The economic intuition behind these expressions is as follows. The firm value (the right hand side of the above equations) comes from three components: i) current profit flow, ii) capital gains from changes in market beliefs (that affect future profit flows), and iii) the potential capital gains or losses from changes in privately known quality.

Naturally, the value of quality determines the seller's investment incentives. In particular, the firm's optimal investment policy is:

$$a(p) = \begin{cases} 0 & \text{if } \lambda D(p) < k \\ 1 & \text{if } \lambda D(p) > k, \end{cases}$$

and any a is optimal when $\lambda D(p) = k$, because the net present value of the investment is zero at that point. Note that due to our technological assumptions, the firm's investment incentives are independent of the state θ: investment increases the probability of a positive shock when the state is low and reduces the probability of a negative shock when the state is low, but in both cases the marginal benefit of investment is the same. This symmetry of investment allows us to write the equilibrium investment strategy as a function of market beliefs alone, $a(p)$.

Trivially, when the firm cannot communicate quality to the market, the value of quality is zero, $D(p) = 0$, leading to zero investment, $a = 0$. By contrast, if the information about quality were public, the firm would fully internalize the benefit of investment, leading to first best levels (i.e., $a = 1$). So unlike standard disclosure models (such as Dye (1985); Jovanovic (1982)) information has social value; it allows the firm to sustain investment and maintain a high level of quality. This is thus precisely the setting where certification could play a positive role by improving investment efficiency. Indeed, in static settings, Albano and Lizzeri (2001) demonstrate that certification plays a positive role, even when the certifier has monopoly power. We next show that this result does not hold in our (dynamic) setting even when the certification cost is arbitrarily small, at least as long as certification is based on current reputation.

To understand the link between the certification strategy and the investment incentives, observe that the evolution of the value of quality when the firm is not certifying is given by

$$rD(p) = \lambda(\tilde{a}(p) - p)D'(p) - \lambda D(p). \tag{1}$$

Let $p_c = \sup\{p \geq 0 : d(p_t, H) = 1\}$ be the highest reputation at which the high type decides to certify and let $\tau_c = \inf\{t > 0 : p_t = p_c, p_0 = 1\}$ be the time that it takes to reach this reputation. Since $\dot{p}_t = \lambda\big(\tilde{a}_t - p_t\big)$, we can integrate (1) over time to get that for any $t \in [0, \tau_c]$, or equivalently for any $p \in [p_c, 1]$, the value of quality at time t is:

$$D(p_t) = e^{-(r+\lambda)(\tau_c - t)} D(p_c). \tag{2}$$

So the value of quality increases as beliefs deteriorate following the last certification. Certification has long lasting effects on the firm reputation because quality is persistent. In turn, the firm has the weakest incentive to invest right after it certifies high quality (something an observer may call “resting on its laurels”).

Furthermore, at the time/belief the firm certifies the value of quality is:

$$D(p_c) = V_H(p_c) - V_L(p_c) = V_H(1) - c - V_L(p_c).$$

Naturally, if the seller does not certify at time $t = \tau_c$, then the market infers that quality is low $\theta_{\tau_c} = L$, and, as a consequence, beliefs drop to zero and remain at that level until the firm re-certifies. Therefore, $V_L(p_c) = V_L(0)$. Moving on to the certification strategy, note that the firm has incentives to certify high quality whenever

$$V_H(p) < V_H(1) - c,$$

namely when the capital gain caused by certification outweighs the (lumpy) certification cost.

Our first lemma, shows that any equilibrium with positive benefits of certification can be characterized by two thresholds p_a and p_c such that the firm never invests before the certification time.[7]

Lemma 1. *Any (pure Markov-Perfect in beliefs) equilibrium is equivalent to an equilibrium defined by two thresholds p_a and p_c such that: $p_a \leq p_c$, $a(p) = 0$ if $p > p_a$ and $d(p, \theta) = \mathbf{1}_{\{p \leq p_c, \theta = H\}}$.*

This is a stark result. It implies that in any equilibrium where certification strategy is contingent on current beliefs, the firm either never invests in quality or only invests when the quality is at the lowest level. We provide a detailed proof in the Appendix, but here is the economic intuition. Suppose the firm has just certified so $p = 1$. If the firm is expected to fully invest in quality at some belief p_a, before the belief drops to p_c (i.e. if $p_a > p_c$), then the market belief would never cross p_a (recall that $\dot{p}_t = \lambda\big(\tilde{a}_t - p_t\big)$). But if so, the market belief would never drop to the certification threshold and we get a contradiction: a firm that is never expected to certify has no incentives to invest at all. [8]

With this result at hand, we can characterize the equilibria further. Since $V_L(0)$ equals the discounted expected gain derived from a positive quality shock, net of both the investment costs required to enable such a shock and the certification expense required to communicate to the market that quality increased, we have

$$V_L(0) = \frac{\lambda a(0)(V_H(1) - c) - a(0)k}{r + \lambda a(0)}.$$

If $p_c > 0$ (so that there is certification in equilibrium), then since failing to certify at p_c makes the market update that the quality is low, $V_H(p_c) = V_H(0) = V_H(1) - c$. Therefore,

[7] Formally, we say that two equilibria $(\tilde{a}, \tilde{d})$ and (a, d) are equivalent if $(\tilde{a}_t, \tilde{d}_t, \tilde{\theta}_t) = (a_t, d_t, \theta_t)$ a.s., each t, where $\tilde{\theta}$ and θ are the quality processes induced by the investment strategies $\tilde{a}$ and a, respectively.

[8] As we show in the proof, even if the firm at p_a chooses an interior level of investment by (2) at slightly lower beliefs it would have strict incentives to put full investment, leading to the same contradiction

the value of quality at $p = p_c$ is

$$D(p_c) = D(0) = \frac{r\big(V_H(1) - c\big) + a(0)k}{r + \lambda a(0)}.$$

Combining it with equation (2) we get the value of quality at any point on the equilibrium path: given by

$$D(p_t) = e^{-(r+\lambda)(\tau_c - t)} \frac{r\big(V_H(1) - c\big) + k}{r + \lambda}.$$

These expressions allow us to fully characterize the set of equilibria for different ranges of the certification cost, c.

Proposition 1. *The set of (pure, belief-contingent Markov-Perfect) equilibria is characterized as follows:*

(i) If $c < \frac{1}{r+\lambda} - \frac{k}{\lambda}$, then, there is an interval $\mathcal{P}_c = [p_c^-, p_c^+]$ of equilibrium certification thresholds. The lower threshold is given by

$$p_c^- \equiv \left[1 - \frac{c}{\frac{1}{r+\lambda} - \frac{k}{\lambda}}\right]^{\frac{\lambda}{r+\lambda}},$$

and the upper threshold is the unique equilibrium threshold in which the zero profit condition $V_H(1) = c$ holds.

In any equilibrium with $p_c > p_c^-$ the firm never invest, that is $a(p_t) = 0$. On the other hand, when $p_c = p_c^-$ we have that for any $a^* \in [0, 1]$, there is an equilibrium in which the high quality seller certifies whenever $p_t \le p_c^-$ and invests $a(p_t) = a^* \mathbf{1}_{\{p_t = 0\}}$. The seller's payoffs are the same in all the equilibria with positive investment and are given by

$$V_L(p_c) = 0$$

and

$$V_H(1) = \frac{k}{\lambda} + c.$$

(ii) If $\frac{1}{r+\lambda} - \frac{k}{\lambda} \le c \le \frac{1}{r+\lambda}$, then the firm never invests and there is an interval $\mathcal{P}_c = [p_c^-, p_c^+]$ such that for any $p_c \in \mathcal{P}_c$ there is an equilibrium such that a high quality firm certifies whenever $p_t \le p_c$. The equilibrium with $p_c = p_c^+$ is the unique equilibrium in which the zero profit condition $V_H(1) = c$ holds, while $p_c = p_c^-$ is the unique equilibrium in which the smooth pasting condition $V_H'(p_c) = 0$ holds.

(iii) If $c > \frac{1}{r+\lambda}$ there is a unique equilibrium in which the firm neither invests nor certifies.

The equilibrium taxonomy depends on the cost of certification. Naturally, for very high values of c, the equilibrium entails no disclosure and hence zero investment. When costs are intermediate, there is certification, but no investment. The most interesting case, however, is when the costs are low, so in what follows we assume that c is low enough that positive investment can be supported. Specifically, we assume that $c < \frac{1}{r+\lambda} - \frac{k}{\lambda}$.

The most remarkable insight of Proposition 1 is that certification is practically unable to mitigate the firm's under-investment problem. Even in the equilibria that have the most investment in quality, the return to investment is zero (when the firm makes the investment in quality it is indifferent between putting positive investment and zero investment). Moreover, investment in quality happens only when the firm is known to have the lowest possible quality.

As the equilibrium analysis shows, access to certification leads, ex post, to excessive certification expenses. Namely, all the benefits of certification stemming from higher investment efficiency are fully offset by the excessive certification expenses that the firm incurs. The firm suffers thus from a sort of "certification intemperance" or "(over) certification trap." Ex ante, the firm would like to commit to no certification (or, as we show in the next section, even better to rare certification), but ex post, when its true quality is high but the market reputation is relatively low, the firm cannot abstain from certification as a means to correct the market's undervaluation. This detrimental effect of certification to the high-quality firm ex-ante profits and to the overall investment in quality stands in contrast with that arising in static settings. For example, Albano and Lizzeri (2001) show that certification generally expands quality investments and increases firm value, even when the certification cost is chosen (at the expense of the firm) by a profit-maximizing monopoly certifier.

This leads to the second stark result in the above proposition. The ex-ante payoff of the high-quality firm in any equilibrium with positive investment is increasing in the certification costs: the firm is better off when the certification is more expensive!

We now provide economic intuition behind the results. First, we have argued before that the firm will at most invest in quality when its reputation is the lowest. But why is the return to investment at that point zero? The reason is that if the firm would have strict incentives to invest in quality at $p = 0$, then it would also have strict incentives to invest before reaching p_c (since $D(p_c) = D(0)$ and $D(p)$ is continuous in p for $p > p_c$). But then we would get the same contradiction as in the previous lemma: the market beliefs would never reach the certification threshold and the firm would actually have no incentives to invest.

Second, this indifference implies $V_L(0) = 0$: since the firm has at most weak incentives to invest in quality at $p = 0$, its equilibrium payoff can be computed by using the strategy of never investing.[9]

To illustrate further the "certification trap," we compare the equilibrium payoffs to the payoffs the firm would obtain if there was no access to certification (or if the firm could credibly commit not to certify). In the absence of certification the firm value is

$$V_H^{nc}(p_t) = V_L^{nc}(p_t) = \int_t^{\infty} e^{-r(s-t)} p_s ds = \frac{p_t}{r+\lambda}.$$

Using the value function obtained in Proposition 1 for the equilibria with investment, we find that

$$V_H(p_t) - V_H^{nc}(p_t) = \begin{cases} -\left(\frac{p_c}{p_t}\right)^{\frac{r+\lambda}{\lambda}} \left(\frac{p_t}{r+\lambda} - \frac{k}{\lambda}\right) & \text{if } p_t \geq p_c \\ \frac{k}{\lambda} - \frac{p_t}{r+\lambda} & \text{if } p_t < p_c. \end{cases} \tag{3}$$

Simple inspection of equation (3) reveals that there is some $\hat{p}$ such that a high quality firm is better off without access certification if its reputation is above $\hat{p}$. Therefore, if a firm is known to have a high quality, it would like to commit to never certify, but if the firm has high quality that is under-appreciated by the market, it would prefer to have access to certification, even if it means that it has to continue certifying in the future.

One can also use our characterization of equilibria to revisit the natural question of pricing of certification. Focus on the equilibria with the most efficient investment. From the point of view of the firm, cheaper certification is offset by the equilibrium effect that the market expects it to certify more often and the second effect dominates, making the firm worse off as c decreases. A profit-maximizing certifier faces a downward-sloping demand curve: lower c leads to more frequent certification. If the marginal cost of the certifier is close to zero (the cost of providing additional certification), we expect the optimal price to be very low. To see this, consider the extreme case of zero marginal cost. Then, as c goes down, certification and hence investment are more frequent. Since paying c is just a transfer, the overall efficiency increases. At the same time, the profits of the firm go down, which implies that the profit of the certifier goes up as well. Hence the certifier profits go up as c decreases towards zero (the limit revenues are positive since the frequency of certification

[9] Another somewhat surprising aspect of the equilibria with investment is that the high-quality firm value at $t = 0$ increases in k. The intuition is as follows. The frequency of certification must be high enough to dissipate enough profits so that $V_H(1)$ is low enough that the L type is indifferent between investing and not investing at $p = 0$. The higher k, the less attractive is investment to the low type, so the certification needs to be less frequent to keep it indifferent (notice that p_c decreases in k). That helps the high type.

goes to infinity). This tendency to set low fees to benefit from more frequent certification adds a new consideration to our standard intuition from the static model in Lizzeri (1999).

4 Time-Contingent Certification

In this section we analyze a second class of equilibria, in which the firm certification strategy, d, depends on time since last certification (rather than the current beliefs, as in the previous section). Our motivation for studying such equilibria is twofold. First, many voluntary certifications have limited duration and firms are expected to certify over relatively regular intervals.[10] Second, since finite-duration certifications are common, we want to understand if such institution/equilibria can avoid the certification trap we characterized in the previous section and result in more efficient investment.

A time-contingent equilibrium is characterized by a number τ that represents the market belief about the duration of the certificate on the equilibrium path. That is, we consider an equilibrium in which after the firm certifies at time t_0, the equilibrium prescribes no certification before time $t_0 + \tau$, and certification with probability one at time $t_0+\tau$, if the firm still has high quality. If the firm has low quality, the equilibrium certification strategy is to certify (after $t_0+\tau$,) as soon as the quality improves.[11]

Our goal is to i) study whether τ improves investment efficiency and ii) characterize the optimal duration τ that can be sustained in equilibrium.

Since between certifications equilibrium beliefs are a deterministic function of time, for every belief-contingent equilibrium we characterized in the previous section in which the high quality firm certifies in intervals of τ_c, there exists an outcome-equivalent time-contingent equilibrium where $\tau = \tau_c$. To focus on equilibria with more investment than in the previous section, we restrict attention equilibria with τ larger than τ_c, where from now on we define $\tau_c \equiv \frac{-\log p_c^-}{\lambda}$ as the amount of time that elapses before beliefs reach the certification threshold p_c^- in the most-efficient belief-contingent equilibrium characterized in Proposition

[10] For example, ISO 9000 expires every three years as does the HMO certification we discussed in the introduction.

[11] Since we keep τ fixed, the equilibria we study are stationary on the equilibrium path. An important off-equilibrium history in these equilibria is when the firm re-certifies sooner than expected, that is, before τ since last certification. If the continuation equilibrium has the same τ as on-path, this equilibrium is stationary also off-path and we would call this equilibrium a time-contingent Markov-Perfect equilibrium. However, we allow the market expectations about frequency of certification to change after the deviation, which enlarges the set of equilibrium outcomes. At the end of this section we discuss when a given τ can be supported by a Markov-Perfect equilibrium.

1. Moreover, we focus on equilibria in which the low-quality invests when the beliefs are the lowest and maintain the assumption that $c < \frac{1}{r+\lambda} - \frac{k}{\lambda}$.

To analyze these time-contingent equilibria, we first consider the firm's investment incentives for a fixed τ. Since the equilibrium is stationary (on path), without loss of generality we reset the time clock to $t_0 = 0$ when the firm certifies high quality. To avoid confusion, since the state variable is different in this section than in the previous one, we introduce new notation: we denote the value function and value of quality as $U_\theta(t)$ and $\bar{D}(t)$, where t is the time since last certification and we write the investment strategy as a_t.

Analogously to our reasoning in the previous section, at time $t < \tau$ the firm's investment incentives depend on

$$\bar{D}(t) = e^{-(r+\lambda)(\tau-t)}\bar{D}(\tau). \tag{4}$$

In any (time-contingent) equilibrium, the firm invests at time t if and only if $\lambda\bar{D}(t) \geq k$, so the optimal investment strategy is also time-contingent. Equation (4) implies that $\bar{D}(t)$ is increasing, so that investment must be a non-decreasing function of time. In other words, the firm's investment strategy, defined as a function of time, must take the form $a_t = \mathbf{1}_{t>\tau_a}$ for some threshold $\tau_a \leq \tau$, where $\tau_a = \tau$ indicates that the firm never invests.[12]

Next we compute the firm's continuation value U_θ in several steps: first, we compute the continuation value at expiration, namely at $t = \tau$, then we determine τ_a as a function of continuation payoffs, then work backwards to obtain the continuation value for $t < \tau$, and finally solve a fixed-point problem to determine τ_a and the continuation payoffs.

Since we are looking at equilibria in which the low-quality firm invests at time $t = \tau$ (and thereafter until the realization of the first positive shock) its continuation value is

$$U_L(\tau) = \frac{\lambda(U_H(0) - c) - k}{r + \lambda},$$

which means that the value of quality at time t is

$$\bar{D}(t) = e^{-(r+\lambda)(\tau-t)}\bar{D}(\tau) = e^{-(r+\lambda)(\tau-t)}\frac{r(U_H(0) - c) + k}{r + \lambda}. \tag{5}$$

This allows us to pin down the firm's investment strategy, namely the time τ_a at which the firm starts investing. The firm is indifferent between investing and not at $t = \tau_a$ if the return

[12] Optimal investment strategy at $t = \tau_a$ is not uniquely determined, but since the firm reaches τ_a over a zero measure of all the times, this has no impact on total payoffs. Hence, when we describe equilibria, we ignore this indeterminacy.

to investment is zero, i.e., if τ_a satisfies:

$$e^{-(r+\lambda)(\tau-\tau_a)}\frac{r(U_H(0)-c)+k}{r+\lambda}=\frac{k}{\lambda}. \tag{6}$$

Solving for τ_a yields

$$\tau_a=\tau+\frac{1}{r+\lambda}\log\left(\frac{r+\lambda}{\lambda}\frac{k}{r(U_H(0)-c)+k}\right). \tag{7}$$

Of course, equation (7) is valid for $\tau_a \in [0,\tau]$. A straightforward computation shows that $\tau_a > 0$ if and only if the return to investment is negative at $t = 0$, namely $\bar{D}(0) < k/\lambda$. If this condition does not hold, then the equilibrium entails first-best investment, $\tau_a = 0$. On the other hand, $\tau_a \leq \tau$ if and only if the return to investment at time τ is strictly positive or, $\lambda(U_H(0) - c) - k > 0$. In words, the firm is willing to invest prior to τ if the return to investment at time τ is strictly positive.

The next step is to compute the firm value during the investment interval, $t \in [\tau_a, \tau)$. Because there is no certification during this interval, the firm value consists of two components: the present value of the cash flows earned through $[t, \tau)$ and the value of the firm at time τ net of the certification cost that will be incurred at that time:

$$U_H(t)=\int_t^\tau e^{-r(s-t)}(p_s-k)ds+e^{-r(\tau-t)}(U_H(0)-c), \tag{8}$$

where p_t evolves according to $\dot{p}_t = \lambda(1 - p_t)$, (since $a_t = 1$ in that interval). Using p_a as the initial belief in the interval $[\tau_{a,}\tau)$, we obtain

$$p_t=1-e^{-\lambda(t-\tau_a)}(1-p_a).$$

Using the definition of $\bar{D}(\cdot)$ and equation (5), we get that the low-quality firm value for $t \in [\tau_a, \tau)$ is

$$U_L(t)=U_H(t)-e^{-(r+\lambda)(\tau-t)}\bar{D}(\tau). \tag{9}$$

The final step in the construction of the value functions requires us to consider $t \in [0, \tau_a]$, when the firm is not investing. Given that there is no investment during this interval, beliefs are $p_t = e^{-\lambda t}$ so the continuation values are

$$U_L(t) = \int_t^{\tau_a} e^{-r(s-t)} p_s ds + e^{-r(\tau_a - t)} U_L(\tau_a) \tag{10}$$

$$U_H(t) = \int_t^{\tau_a} e^{-r(s-t)} p_s ds + e^{-r(\tau_a - t)} U_L(\tau_a) + e^{-(r+\lambda)(\tau_a - t)} \bar{D}(\tau_a). \tag{11}$$

Notice the asymmetry between the two states: since in this time interval the firm is not investing, it can experience a negative shock in the high state but no shocks in the low state.

We can now pin down the investment threshold τ_a using equation (10), along with (7) and the optimality condition $\bar{D}(\tau_a) = k/\lambda$. At the same time we can pin down the equilibrium payoffs as a solution to a fixed-point problem and establish existence of equilibria. We summarize this in the following proposition:

Proposition 2. *For any τ and τ_a let*

$$v(\tau, \tau_a) \equiv \frac{\int_0^{\tau} e^{-r(s-t)} p_s ds - \left(e^{-r\tau_a} - e^{-r\tau}\right)\frac{k}{r} + \left(e^{-(r+\lambda)\tau_a} - e^{-r\tau_a}\right)\frac{k}{\lambda} - c}{1 - e^{-r\tau}} \tag{12}$$

$$g(\tau, \tau_a) \equiv \frac{k}{\lambda}\frac{r+\lambda}{r} e^{(r+\lambda)(\tau - \tau_a)} - \frac{k}{r}. \tag{13}$$

Let $\tau_a \in (0, \tau)$ be a solution of $v(\tau, \tau_a) = g(\tau, \tau_a)$. Then, there is a (time-contingent) equilibrium with $a_t = \mathbf{1}_{\{t > \tau_a\}}$. In addition, if $v(\tau, 0) \geq g(\tau, 0)$ there is an equilibrium with $\tau_a = 0$. The high-value firm ex-ante equilibrium payoff is $U_H(0) = v(\tau, \tau_a) + c$. Finally, for every $\tau > \tau_c$ there exists at least one equilibrium; and for any equilibrium there is positive investment before τ .

As we discussed above, if $\tau_a \in (0, \tau)$, the equilibrium beliefs/reputation are non-monotone between two certifications (see Figure 2 below). The market rationally expects that the firm is shirking right after certification, so the beliefs go down close to $t = 0$. Yet, as the expiration date approaches, the firm has incentives to invest again. The market rationally foresees that and beliefs start going up. Hence, certification happens not when the firm reputation is the lowest, but after it rebounds. For any $\tau > \tau_c$ the only other possibility is that in equilibrium $\tau_a = 0$ and reputation never drops. So generally, our model predicts the following pattern of beliefs and certification. If the high-quality firm reaches τ, certification happens either at the highest reputation or after reputation has recently improved. If the low-quality firm reaches τ, it fails to certify, reputation discontinuously drops, and the firm certifies again after it regains high quality.

To see the intuition for why $\tau > \tau_c$ results in strict incentives to invest in quality, recall that if $\tau = \tau_c$, the firm is just indifferent to invest at $t = \tau_c$. A less frequent certification implies a strictly higher expected profit of the high quality firm, even if it is not expected to put any investments. That makes the firm prefer to invest before reaching τ. As the market rationally expects this (efficient) investment, that makes the payoff of the high-quality firm even higher, which further reinforces the incentives to invest in quality.

For a fixed duration τ, there are sometimes multiple τ_a that are consistent with equilibrium according to the system of equations in Proposition 2. This is caused by strategic complementarity of reputation and investment: pessimistic beliefs about the firm's investment policy reduce the payoffs from certification and that in turn reduces incentives to invest (and vice versa). By reducing the return to investment, low investment levels may then become a self-fulfilling prophecy. We defer the analysis of multiplicity to Section 4.1, but for now we define

$$\mathcal{E}(\tau) \equiv \{t_a \in [0, \tau] : v(\tau, t_a) \geq g(\tau, t_a) \text{ and } (v(\tau, t_a) - g(\tau, t_a))t_a = 0\},$$

as the set of equilibrium investment thresholds, given duration τ, and we let $\underline{\tau}_a = \inf \mathcal{E}(\tau)$ and $\overline{\tau}_a = \sup \mathcal{E}(\tau)$ be the lower and higher investment thresholds that can be supported in equilibrium.

With this definition, we can further characterize time-contingent equilibria:

Proposition 3. *Let $U_H(0|\tau, \tau_a)$ be the high-quality firm's ex-ante expected (time-contingent) equilibrium payoff when the certification duration is τ and the equilibrium investment threshold is τ_a. Then*

(i) There is some finite $\tau > \tau_c$ such that $U_H(0|\tau, \tau_a) > V_H^{nc}(1)$ for all $\tau_a \in \mathcal{E}(\tau)$.

(ii) $\underline{\tau}_a$ and $\overline{\tau}_a$ are monotone nondecreasing in c.

(iii) Let $\underline{U}_H(0|\tau) := U_H(0|\tau, \underline{\tau}_a(\tau, c), c)$ and $\overline{U}_H(0|\tau) := U_H(0|\tau, \overline{\tau}_a(\tau, c), c)$ be the ex-ante expected payoff in the equilibrium with minimum and maximum investment threshold, respectively. Then, $\sup_{\tau \geq \tau_c} \underline{U}_H(0|\tau)$ and $\sup_{\tau \geq \tau_c} \overline{U}_H(0|\tau)$ are decreasing in c.

Let us discuss these results. In Section 3 we have shown that under the belief-contingent equilibrium the high-quality firm (that is known to be high-quality at $t = 0$) would be better off by committing to no-certification. By contrast, Proposition 3(i) shows that there exists a duration of the certification that generates even better payoffs, no matter what equilibrium

τ_a the firm and the market coordinate on. Therefore, if the duration of the certification is chosen optimally, we can overcome the paradoxical result in the previous section that certification does not promote investment and hurts the firm.

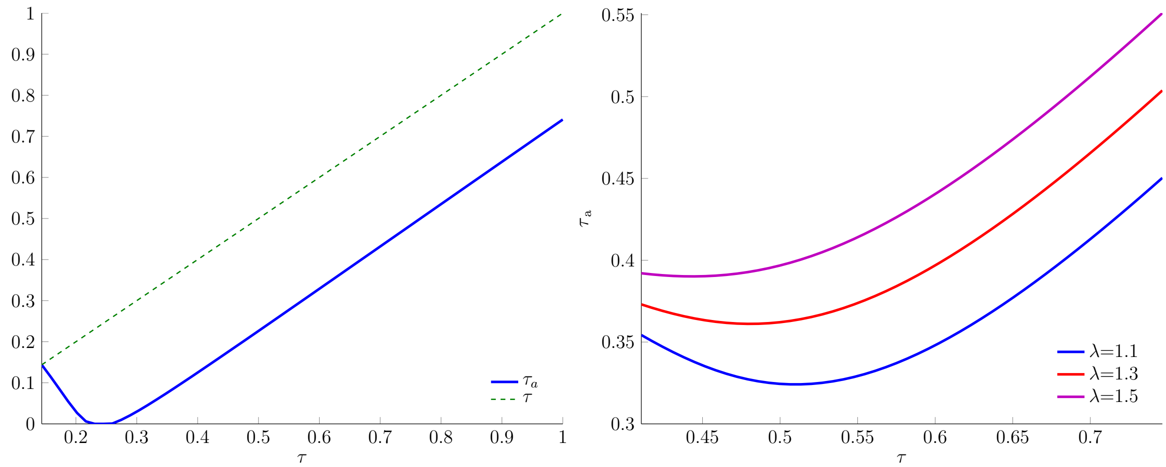
The optimal duration τ is determined by the following trade off. As τ changes, there are two equilibrium effects. First, longer τ means less certification costs, which increases firm payoffs and hence increases incentives to invest in quality close to τ. On the other hand, a longer τ can mean that the firm has such a long time till next certification that it chooses to shirk right after certification. This trade off is such that the optimum is always interior; neither no certification nor certification at the frequency of the belief-contingent certification are optimal.

The intuition behind Proposition 3(ii) is that the lower c, keeping the frequency of certification fixed, increases the high-quality firm payoffs, increasing incentives to invest in quality, which means that the firm starts investing sooner (and rational expectations by the market reinforce this effect). Finally, Proposition 3(iii) states that firms with high quality benefit from lower certification costs in time-contingent equilibria with the optimal τ. That resolves another paradox of the belief-contingent equilibria we discussed in the previous section.[13]

Perhaps surprisingly, investment in quality is non-monotone in the duration of certification. In particular, less-frequent certification can speed-up investment in quality. We illustrate it in panel a) of Figure 1. The horizontal axis traces different equilibrium τ starting at τ_c which we know results in $\tau_a = \tau_c$. As we can see in panel a), as τ initially increases above τ_c, τ_a rapidly declines and in this example reaches $\tau_a = 0$ for a range of τ, for the reasons we discussed above (less spending on certification increases the value of reputation). Only when τ gets sufficiently high, τ_a starts increasing/investment gets delayed as τ increases.

An interesting comparative static is on how equilibrium investment depends on the volatility of the firm's environment, as represented by λ. It might seem that the return to investment should increase in λ even for fixed τ, because the role of investment is prevent negative shocks. Indeed, under symmetric information the return to investment increases in λ. However, this argument ignores a second effect: an increase in λ reduces the persistence of quality, weakening the incentives to invest. Our numerical examples indicate that the latter effect tends to dominate. Figure 1b shows examples where a higher λ results in less investment (higher τ_a for any τ).

[13]This last comparison is somewhat complicated since even for the optimal τ there may be multiple time-contingent equilibria with different τ_a, so we compare equilibria with the lowest and the highest selections of the equilibrium investment.



(a) Investment threshold τ_a as a function of τ ($k = 0.2$, $c = 0.075$, $\lambda = 1$, $r = 1$, $\tau_c = 0.178$).

(b) Volatility and Investment ($k = .2$,$c = .2$,$r = .5$).

Figure 1: Investment threshold.

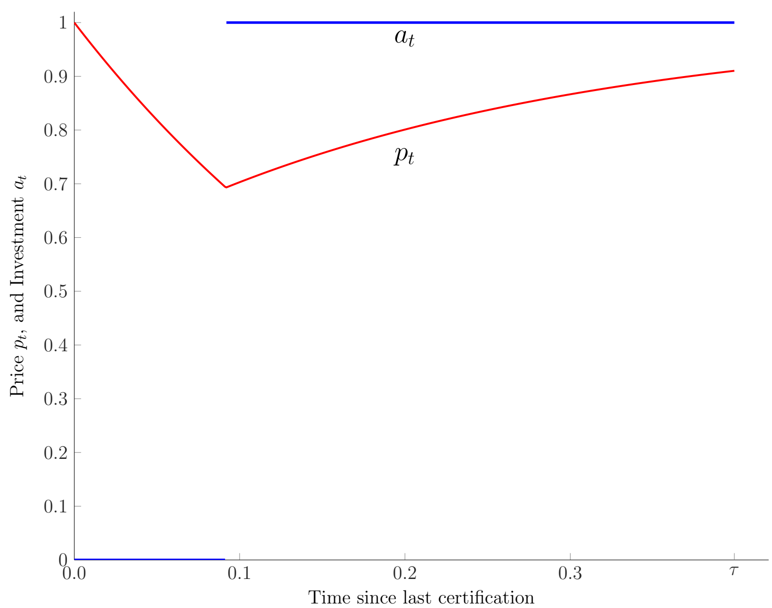


Figure 2: Investment and Price Path. Parameters: $k = 0.5$, $c = 0.05$, $\lambda = 4$, $r = 0.1$, $\tau = 0.39$

Figure 3 shows how the cost of certification affects the best equilibrium, that is, how (τ, τ_a) that maximize $U_H(0)$ change with c. The example illustrates that, under time-contingent certification, the level of investment and the firm payoffs decrease as the cost of certification goes up, which is the opposite effect to what we described for belief-contingent certification. The discontinuity in $U_H(0)$ and τ_a correspond to high enough c so that it is no longer possible to maintain the efficient effort. Note also that the optimal τ is non-monotone in c.

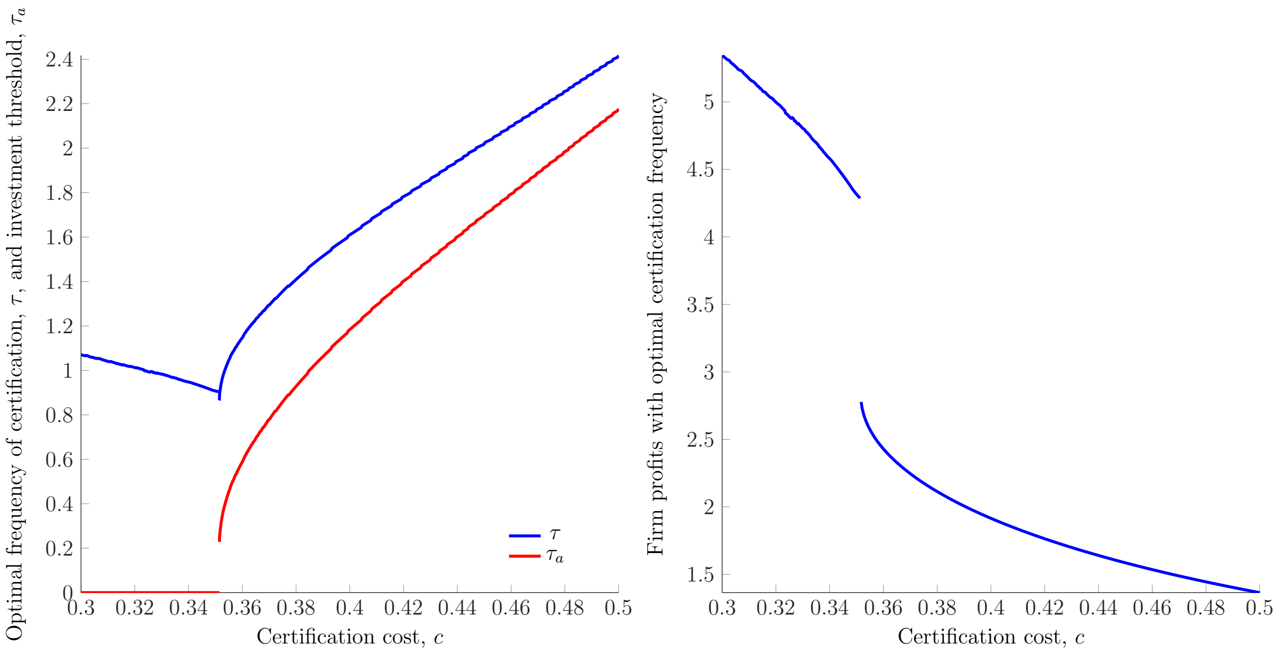
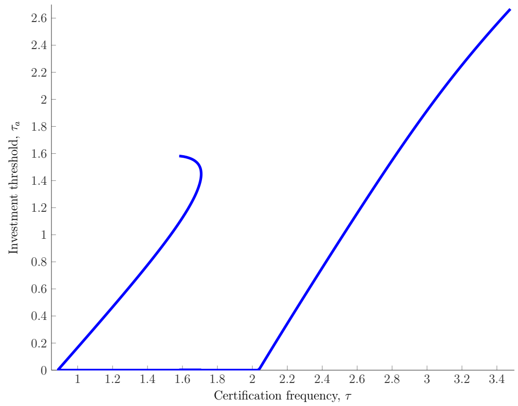
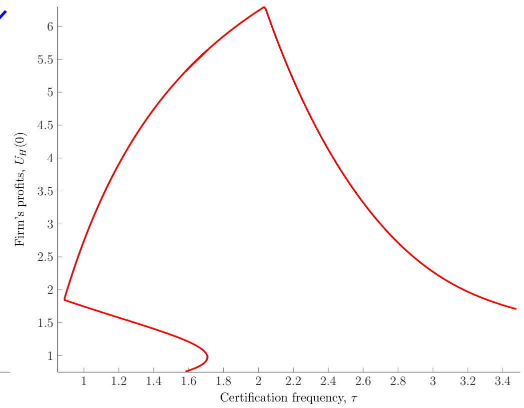


Figure 3: Comparative statics best equilibria as cost of certification varies. The best equilibria is the one that solves the problem $\max_{\tau\geq 0, \tau_a \in \mathcal{E}(\tau)} \; U_H(0|\tau, \tau_a)$. Parameters: $k = 0.2$, $\lambda = 1$, $r = 0.1$

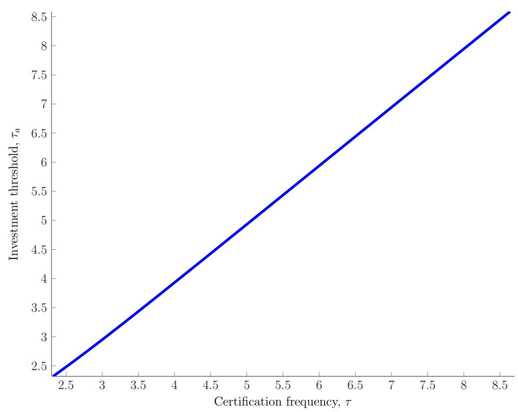
To finish this section, we discuss when the time-contingent equilibrium outcomes we described, can be supported as outcomes of Markov-Perfect equilibria. For simplicity, in our proofs so far we have assumed that if the firm deviates and certifies before τ then the continuation equilibrium is one with τ' short enough that the firm has continuation payoff of exactly c, making this a non-profitable deviation (this τ' corresponds to the threshold belief p_c^+ in Proposition 1). But this is usually much stronger punishment than necessary. For example, in case the equilibrium has $\tau_a = 0$, then the equilibrium outcome can be supported by simply continuing with the same equilibrium, since the deviating firm pays c earlier without getting any reputational benefits. In general, the firm will have no incentives to deviate to earlier certification, even if the continuation equilibrium does not punish it with the expectation of a more frequent certification, if $(U_H(0) - U_H(t) < c)$ for all $t < \tau$, which can be numerically verified (our calculations suggest that for small c and k the equilibria that maximize $U_H(0)$ can be supported as Markov-Perfect equilibria, but that is not generally true).



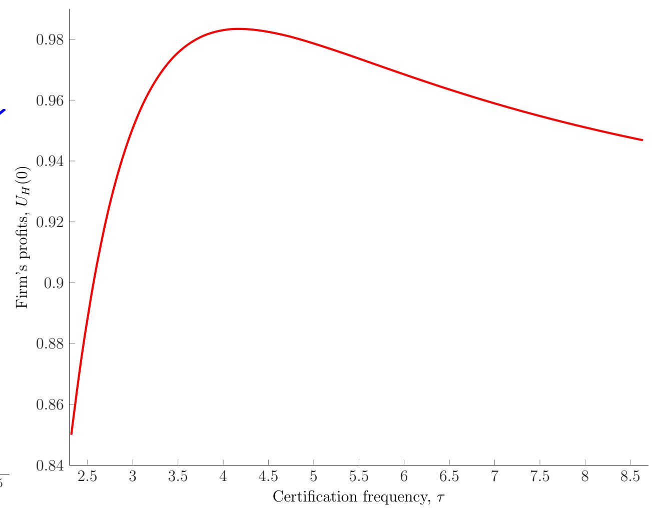
(a) Investment threshold τ_a with low cost of investment, $k = 0.06$.



(b) Firm's profits with low cost of investment.



(c) Investment threshold τ_a with high cost of investment, $k = 0.15$.



(d) Firm's profits with high cost of investment, $k = 0.15$.

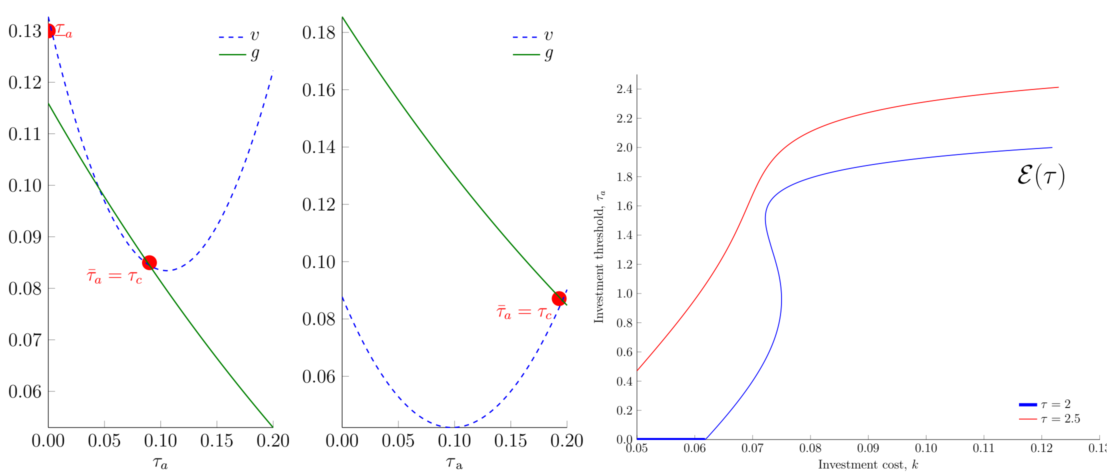
Figure 4: Investment threshold and firm profits. Parameters: $c = 0.7$, $\lambda = 1$, $r = 0.1$. For this set of parameters, the frequency of certification in the best Markov perfect equilibrium is $\tau_c = 1.58$ and $\tau_c = 2.32$ for $k = 0.06$ and $k = 0.15$, respectively.

4.1 Multiplicity

As our game has a signaling component (the privately-informed firm takes public certification actions), it is not surprising that there are many equilibria with different market beliefs about when the firm will certify. This leads to multiple equilibria in Proposition 1 and a large range of τ's that can be supported in time-contingent equilibria.

In addition to the multiplicity of durations, for a fixed duration τ there may be multiple equilibrium investment thresholds τ_a. Figure 4 shows indeed that for low investment cost, k, there may be multiple equilibrium investment thresholds, for some τ, whereas for high k the threshold is unique. As we mentioned above, the multiplicity of investment thresholds is due to the strategic complementarity between reputation and investment. If the market holds relatively negative beliefs about firm investment, that reduces the value of certification and the return on investment. This in turn justifies the market negative beliefs thus creating a self-fulfilling prophecy (this is reminiscent of the statistical discrimination literature (Arrow (1998)).

To further illustrate the multiplicity, Figure 5a shows the equilibrium investment threshold(s) when $\tau = \tau_c$. It shows that when k is relatively low there are three equilibria. One of them implements efficient investment ($\tau_a = 0$), whereas the one with the least investment has no investment before τ. For large k, in contrast, the equilibrium τ_a is unique (no investment before τ). Figure 5b illustrates the equilibrium multiplicity from a different angle. It shows the equilibrium correspondence (the set of τ_a that are consistent with equilibrium for two levels of τ) for different levels of investment cost, k. In this example, for $\tau = 2$, there is a unique equilibrium when k is low; three equilibria for a range of intermediate values of k; and one equilibrium when k is sufficiently high. In contrast, when $\tau = 2$, the equilibrium τ_a is unique for all k.



(a) The left panel shows the case with multiple equilibria, where the smaller investment threshold is $\tau_a = 0$. The right panel depicts the case where the equilibrium is unique and equal to τ_c. Parameters: $\tau = \tau_c$, $c = 0.09$, $\lambda = 1$, $r = 1$. The left panel has $k = 0.07$ while the right panel has $k = 0.12$

(b) The figure shows the equilibrium correspondence τ_a for different cost of investment, k, and duration, τ. Parameters: $c = 0.7$, $\lambda = 1$, $r = 0.1$.

Figure 5: Multiplicity of Equilibria

5 Unobservable Expiration Date

In many industries buyers ignore or do not even observe when a certificate was issued and when it will expire. They may only know whether the firm is currently certified. For example, when we go to a restaurant exhibiting a Michelin star, we typically do not know when this rating will be revised; all we know is that in the current period the restaurant exhibits a Michelin star. This implies that the ability of buyers to update perceptions of quality based on time may be limited in some markets.

We propose an extension of our model to capture this issue. Consistent with the analysis in Section 4, we consider time-contingent equilibria where firms re-certify quality only after a period of length τ since the last certification. Our interpretation is that the rating/certification agency establishes a duration τ and firms must re-certify upon expiration to be allowed to display the credentials. Consumers however, are unaware of the expiration date and base their purchase decision exclusively on the certification status (and knowledge of τ).

Let the stochastic process $R_t \in \{0, 1\}$ take value one if the firm is certified at time t and zero otherwise. Consumers observe R_t but do not observe $\tau - t$ (that is, the time left till expiration). Because investment and expected quality may vary across the certification cycle, consumers must form beliefs with respect to time left for expiration. In general, this is a complex statistical inference problem as we need to specify customer prior beliefs over the time between certification times and they update based on the observed certification status. We can greatly simplify the problem if we look at the long run (steady-state) equilibrium. In this case, we assume that consumers beliefs are given by the ergodic distribution of (θ_t, R_t), which can be computed using the renewal theorem. This is equivalent to assuming that consumers don't observe calendar time and have "improper uniform priors" about it. If we denote the ergodic distribution by $\mathbb{P}^e(R, \theta)$, then the price is given by the conditional probability

$$\mathbb{P}^e(\theta = H | R = 1) = \frac{\mathbb{P}^e(R = 1, \theta = H)}{\mathbb{P}^e(R = 1)}.$$

A realistic feature of this setting is that instead of a continuum of prices there are just two prices in equilibrium: one for uncertified firms and another for certified firms.

Lemma 2. *Suppose that consumers have improper uniform prior beliefs about calendar time.*

Then, the price conditional on certification $R \in \{0,1\}$ *is*

$$p(R) = \begin{cases} 0 & \text{if } R = 0 \\ p_e & \text{if } R = 1 \end{cases}$$

where $p_e \equiv \frac{1}{\tau}\int_0^\tau p_t dt$ *and* p_t *is the belief of an external observer who would know both* R *and* t*.*

Using these prices, we can calculate the profits of the seller at time zero as given by

$$U_H(0) = \int_0^\tau e^{-rt}(p_e - k\mathbf{1}_{t>\tau_a})dt + e^{-r\tau}(U_H(0) - c) + \left(e^{-(r+\lambda)\tau_a} - e^{-r\tau_a}\right)\frac{k}{\lambda}.$$

The following proposition, characterizes the equilibrium with unobservable expiration date.

Proposition 4. *For any* τ *and* τ_a *let*

$$v^u(\tau,\tau_a) \equiv \frac{p_e}{r} - \frac{e^{-r\tau_a} - e^{-r\tau}}{1-e^{-r\tau}}\frac{k}{r} + \frac{\left(e^{-(r+\lambda)\tau_a} - e^{-r\tau_a}\right)\frac{k}{\lambda} - c}{1-e^{-r\tau}}. \tag{14}$$

where

$$p_e = 1 - \frac{\tau_a}{\tau} + \frac{e^{-\lambda(\tau-\tau_a)} - e^{-\lambda\tau}}{\lambda\tau} \tag{15}$$

Let $\tau_a^u \in (0,\tau)$ *be a solution to the equation* $v^u(\tau,\tau_a^u) = g(\tau,\tau_a^u)$*, then, there is an equilibrium with* $a_t = \mathbf{1}_{\{t>\tau_a^u\}}$*. In addition, if* $v^u(\tau,0) \geq g(\tau,0)$ *there is an equilibrium with* $\tau_a^u = 0$ *and* $p_e = 1$*.*

So, as in Proposition 2 the firm's investment policy is given by a threshold τ_a that satisfies the market rational expectations. We can further describe how the firm's investment policy is affected by certification costs and the effect of expiration observability.

Proposition 5. *Let* $\{\underline{\tau}_a, \overline{\tau}_a\}$ *and* $\{\underline{\tau}_a^u, \overline{\tau}_a^u\}$ *be the extreme investment thresholds when the expiration date is observable and unobservable, respectively, and let* $\{\underline{U}_H(0), \overline{U}_H(0)\}$ *and* $\{\underline{U}_H^u(0), \overline{U}_H^u(0)\}$ *be its respective ex-ante payoff. Then,*

(i) The extreme investment thresholds $\underline{\tau}_a^u$ *and* $\overline{\tau}_a^u$ *are monotone nondecreasing in* c*. The ex-ante payoff* $\underline{U}_H^u(0)$ *and* $\overline{U}_H^u(0)$ *are decreasing in* c*. In particular,* $\sup_{\tau\geq\tau_c}\underline{U}_H^u(0)$ *and* $\sup_{\tau\geq\tau_c}\overline{U}_H^u(0)$ *are also decreasing in* c*.*

(ii) There is more investment with observable expiration dates; that is, $\underline{\tau}_a \leq \underline{\tau}_a^u$ *and* $\overline{\tau}_a \leq \overline{\tau}_a^u$*. The ex-ante payoffs are lower with unobservable expiration:* $\underline{U}_H^u(0) \leq \underline{U}_H(0)$ *and* $\overline{U}_H^u(0) \leq \overline{U}_H(0)$*.*

Part (i) states that we have the same comparative statics with respect to c as in the case with observable expiration. Payoffs and investment are decreasing in c so efficiency increases when the cost of certification is lower.

Part (ii) compares the efficiency of investment and the ex-ante payoff of the cases with and without observable expiration. Under unobservable expiration, certification plays again a positive role in terms of promoting investment and increasing firm value. However, being less informative, certification is less effective relative to the case when consumers observe the expiration date. We illustrate these comparisons in Figure 6. The main step in the proof of proposition 5(ii) is to show that for any fixed expiration τ and investment policy τ_a we have

$$\frac{\int_0^\tau re^{-rt}p_t dt}{1-e^{-r\tau}} \geq \frac{\int_0^\tau p_t dt}{\tau},$$

that is the average with weights $re^{-rt}/(1-e^{-r\tau})$ is weakly greater than the simple time average. The previous inequality is obvious if p_t is non-increasing in time; however, p_t can be non-monotone, decreasing just after certification and increasing just before expiration. The left hand side puts more weight in the price just after certification (which is higher than p_e) and less weight in the price just before expiration (which might be higher or lower than p_e). It turns to be the case that, no matter the value of τ and τ_a, the higher weight on high prices just after certification always dominates. This is true for any fixed $\tau_a \leq \tau$ but the equilibrium τ_a's need not be the same in the two models. However, if for a fixed policy the present value of revenues is higher with observable expiration, then the value of quality must be higher and we must have more investment. This second effect reinforces the benefit of observing expiration. More formally, if we let p_t be the probability of high quality given an investment policy τ_a and p_t^u be the one with an investment policy τ_a^u, then we have that

$$\frac{\int_0^\tau re^{-rt}p_t dt}{1-e^{-r\tau}} \geq \frac{\int_0^\tau re^{-rt}p_t^u dt}{1-e^{-r\tau}} \geq \frac{\int_0^\tau p_t^u dt}{\tau} = p_e.$$

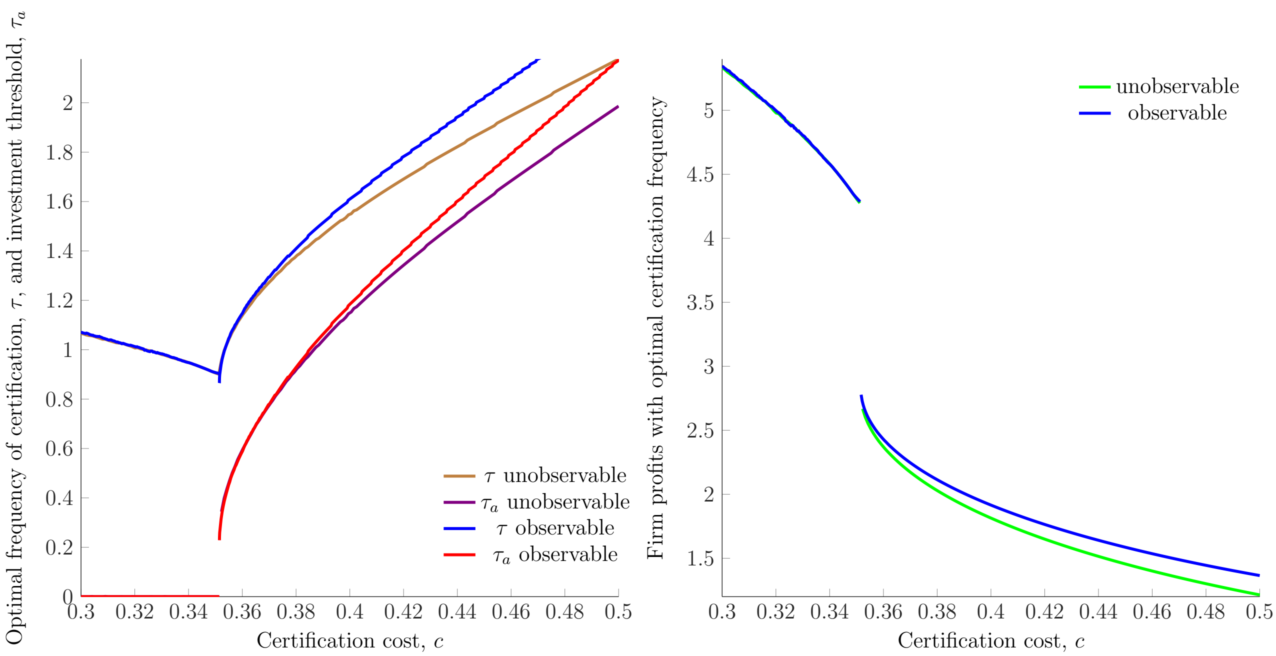


Figure 6: Comparative statics best equilibria as cost of certification varies. The best equilibria is the one that solves the problem $\max_{\tau\geq 0,\tau_a\in\mathcal{E}(\tau)}\ U_H(0|\tau,\tau_a)$. Parameters: $k=0.2$, $\lambda=1$, $r=0.1$

6 Discussion, Interpretation and Empirical Implications

As we discussed in the Introduction, certification is prevalent in modern markets. For example, Wikipedia notes that "over one million organizations worldwide are ISO certified, making ISO one of the most widely used management tools in the world today."[14] Certification is undertaken by universities, restaurants, HMOs, and professional workers (e.g., physicians, lawyers, teachers, and accountants). Firms certify their products as environmentally friendly, kosher, organic, or humane raised. They even certify management practices through the ISO system, designed for "companies who want to ensure that their products and services consistently meet customer's requirements, and that quality is consistently improved."[15]

Even though in the model we describe the cost of certification as a fee paid to the certifier (as in Jovanovic (1982)), we interpret certification costs as consisting of the firm resources consumed by certification, including the certifier fees, and the costs associated with the preparation and dissemination of the information. Alternatively, the certification cost can be interpreted as the proprietary cost of disclosing the information, in the spirit of Verrecchia

[14] *en.wikipedia.org/wiki/ISO_9000* (accessed on 7/8/2015).

[15] *www.iso.org/iso/iso_9000* (accessed on 7/8/2015).

(1983). For example, disclosing favorable information may trigger competition in the product market (Darrough and Stoughton (1990)) or litigation risk in the capital market (Francis, Philbrick and Schipper (1994)). It may strengthen the bargaining power of the firm's labor unions, or accelerate rate of return regulations in monopoly settings.

Some of our stark results depend on the assumption of linear investment costs. If the costs are somewhat non-linear, but the marginal cost of effort at $a = 0$ is positive and the marginal cost is close to constant, the main result continue to hold: under belief-based reputation we cannot have much investment before beliefs hit p_c since otherwise we would never reach those beliefs, so belief-contingent certification will not be very successful in resolving the investment problem. On the other hand, time-contingent certification, by reducing the frequency and hence total certification costs, is likely to increase the value of reputation and increase investment. Another robust feature of the model is that in any time-contingent equilibrium, the incentives to invest increase as the time to expiration shrinks. Hence, we expect reputation to be typically non-monotone in equilibrium, decreasing soon after certification, when the firm puts little investment, and increasing again before expiration. The results that do change are that some investment is possible before p_c, as long as it is sufficiently small so beliefs continue decreasing. Moreover, we have constructed examples with quadratic costs, in which the best belief-contingent equilibrium is better for the high type than no certification whatsoever. Yet, in such examples time-contingent certification can still perform strictly better. This confirms the intuition from this paper that belief-contingent certification leads to excessive certification and better incentives for investment.

We purposely ignored alternative sources of information that the market may use to learn about quality, notably public ratings (Ekmekci (2011)) and consumer reviews (Cabral and Hortacsu (2010)). By restricting attention to certification as the only information channel, we thus consider a clean setting for understanding the informational role of certification. In our setting information can have social value (since it can help improve investment in quality) and we seek to understand whether and when certification can deliver such value. In many markets certification is the main source of information about quality that the customers have and hence we think our model is applicable to such markets. In other markets customers learn both from reviews (or other outside news) and from voluntary certification. To understand such markets better, we think leave for future research analysis of a model with a combination of these sources of information.

6.1 Empirical Implications

A central question in economics is why firm performance persists. Empirical evidence suggests that the degree of persistence in firm performance varies widely from industry to industry (Mueller, 1977; Jacobsen, 1988; McGahan and Porter, 1999). In our model, the persistence of profits depends upon several factors: the cost of certification, the profitability of investment opportunities, the obsolescence rate of assets, and the extent to which assets are subject to technology shocks. Similar firms display heterogeneity in performance and product quality, which is persistent and positively correlated.

When is certification helpful? Some certification systems have been criticized. Despite widespread use, the ISO process has been criticized as wasteful. Dalgleish (2005) cites the "inordinate and often unnecessary paperwork burden" of ISO, and asserts "managers feel that ISO's overhead and paperwork are excessive and extremely inefficient. Despite their dislike, many companies are registered. Firms maintain their ISO registration because almost all of their big customers require it." Our model sheds light on this apparent contradiction. Since the mere availability of certificates modifies market beliefs about uncertified firms, it can operate as a threat that destroys firm value by forcing firms to incur large costs to avoid the price penalty the market applies to uncertified firms. Our analysis shows that certification is an effective communication channel in industries where this market threat is not so severe, either because certification is too expensive, consumers are not so aware of the expiration date (see extension in Section 5) or the industry as a whole is able to coordinate in order to mitigate excessive certification.

Quality and investment cycles In many industries shocks to reputation trigger cycles in product quality (see e.g., Cabral and Hortacsu (2010)). Our analysis shows that certification generates quality and investment cycles, with firms shirking right after certification, to *milk* their reputation (Mailath and Samuelson (2001); Liu and Skrzypacz (2014)), and exerting high effort prior to the re-certification date, to increase the chances of passing the certification test.[16] While this observation suggests that average investment/quality should increase if one shortens the certification cycle, we have demonstrated that the opposite could be true when the cost per certification is constant. Quality and investment are ergodic in our setting, unlike the case with bad news in Board and Meyer-ter Vehn (2013). This is because the

[16] Unlike previous reputation models, our model is able to generate high effort even when reputation is at the highest possible level.

possibility of certification allows the firm to break reputation traps that would otherwise lead to a permanent deterioration of product quality and a complete shut-down of investment (as in Cabral and Hortacsu (2010)).

Entry and certification Atkeson, Hellwig and Ordoñez (2014) argue that firm dynamics are often driven by uncertainty regarding the quality of new products: "If it takes buyers time to learn about the quality of entering firms, these firms initially face lower demand and prices until they are able to establish a good reputation for their product." In this paper, we assume the firm is established but the analysis extends directly to settings with entry in which the firm has to evaluate whether to commercialize an innovation or not. Our analysis sheds light on whether certification facilitates or deters entry/innovation. Policy makers often seek to stimulate innovation by reducing entry costs (see e.g., Djankov et al. (2002)). As Atkeson, Hellwig and Ordoñez (2014) argue, in many cases the main barrier to entry is consumer uncertainty about the quality of a new product. To reduce this barrier, policy makers could try to implement low cost certification systems, without realizing that such a policy could lead to stronger market penalties for uncertified firms, a higher intensity of certification and, ultimately, higher certification expenses. Thus, such a policy could end up deterring entry rather than stimulating it. Our paper suggests that regulating certification to have longer duration and observable expiration can be beneficial.

7 Conclusion

This paper studies certification as a mechanism used by firms to improve their reputation when quality and investment are unobservable. Our focus is on certification and investment incentives. Unlike the prior reputation literature, we consider a dynamic setting in which the firm decides not only whether to certify, but also when. Moreover, in our model reputation depends on endogenous and voluntary disclosure instead of exogenous signals (for example, consumer reviews).

We contrast the efficiency of equilibria in which firm strategies (and market expectations about certification) are belief-contingent against equilibria in which certification is time-driven. We also study the effect of unobservable expiration dates (so that customers observe only the certification status but do not know when the certificate was issued or when it expires). Perhaps the main lesson of this paper is that belief-contingent certification has no ability to increase investment and that, paradoxically, it often reduces firm value. Certification leads to higher quality only when it is time-contingent and the frequency is not too high. Moreover, transparency of expiration additionally improves incentives. This can explain why a lot of voluntary certifications have a pre-specified duration and why many make the expiration or granting date public

In terms of empirical predictions we show that, under time-based certification, both investment and reputation exhibit ergodic dynamics, with the firm shirking right after certification takes place –to milk or cash in its reputation– and exerting effort/choosing high investment right prior recertification (even when reputation is highest) to pass the recertification test. In our model, certification determines cycles of quality and reputation and leads to heterogeneity in profitability and persistence in performance. When costs of certification are low, optimal duration may result in the firm always maintaining high quality. In that case, on the equilibrium path the firm always certifies when the time comes and the market beliefs never change, making the certification seemingly wasteful. Yet, the expense is necessary to maintain proper incentives.

Appendix

Proof Lemma 1

Proof. Let $p_a \equiv \sup\{p \in [0,1] : a(p) > 0\}$ and $p_c \equiv \sup\{p \in [0,1] : d(p,H) = 0\}$ and let $\tau_a \equiv \inf\{t > 0 : p_t = p_a, p_0 = 1\}$ and $\tau_c \equiv \inf\{t > 0 : p_t = p_c, p_0 = 1\}$. First, we show that in any equilibrium $p_a \leq p_c$. Looking for a contradiction, suppose that $p_a > p_c$. Let's consider the behavior of beliefs at the threshold p_a. If $a(p_a) \geq p_a$ then $\lambda(a(p_a) - p_a) \geq 0$ so beliefs never cross the threshold p_a. On the other hand, if $a(p_a) < p_a$ then beliefs cross the threshold p_a however if this is the case, we have that $k/\lambda = D(p_a) = e^{-(r+\lambda)(\tau_c - \tau_a)} < e^{-(r+\lambda)(\tau_c - t)} = D(p_t)$ for all $t \in (\tau_a, \tau_c]$. This means that $a(p_a - \epsilon) = 1$ but if this is the case then beliefs can never cross the threshold p_a. This in turn implies that $\tau_c = \infty$, so that $D(p_t) = e^{-(r+\lambda)(\tau_c - t)} D(p_c) = 0$. This contradicts the hypothesis that $p_a > p_c \geq 0$ which requires that $\lambda D(p_a) \geq k$.

Second, we analyze the certification strategy. By definition we have that $d(p,\theta) = 0$ for $p > p_c$ and $d(p_c, H) = 1$. If the firm fails to certify at time τ_c beliefs drop to zero so $p_{\tau_c^+} = 0$. The next step is to specify the certification strategy when $p_0 = 0$. We consider two cases: $V_H(1) - c > 0$ and $V_H(1) - c = 0$ ($V_H(1) - c < 0$ is trivial because in this case certification is suboptimal so $d_t = 0$). Let's consider the case with $V_H(1) - c > 0$ first. Suppose that $\tilde{p} = \inf\{p : d(p,H) = 1\} > 0$ and let $\tilde{\tau} = \inf\{t : p_t = \tilde{p}, p_0 = 0\}$. Using the incentives equation we have

$$D(0) = e^{-(r+\lambda)\tilde{\tau}} D(\tilde{p}).$$

By construction we have that $V_H(\tilde{p}) = V_H(1) - c = V_H(p_c) = V_H(0)$ (Note that it cannot be the case that $V_H(p_c) \neq V_H(0)$ as this would contradict the optimality of the certification strategy). Similarly, we also have $V_L(\tilde{p}) = V_L(0)$ because the market infers that the firm has low quality if it fails to certify when $p_t = \tilde{p}$. Thus, $D(\tilde{p}) = V_H(\tilde{p}) - V_L(\tilde{p}) = V_H(0) - V_L(0) = D(0) = D(p_c)$. Replacing in the incentives equation we get

$$D(0) = e^{-(r+\lambda)\tilde{\tau}} D(0) \Rightarrow D(0) = 0.$$

If this is the case then we have that $a(p) = 0$ for all $p \in [0, \tilde{p}]$ and in particular $a(0) = 0$ so $(p_0 = 0, \theta_0 = L)$ is an absorbing state and $V_L(0) = 0$. This, together with $D(0) = 0$, implies that $V_H(0) = 0$, which contradicts the hypothesis $V_H(1) - c > 0$. Hence, it must be the case that $\tilde{p} \leq 0$.

Next, we consider the case with $V_H(1) - c = 0$. In this case, by a similar argument as the one used before, we have that $D(0) = 0$, so $a(0) = 0$ and $(p_t = 0, \theta_t = L)$ is an absorbing state. This means that for any strategy $\tilde{d}_t$ in which the low quality firm never certifies there is some threshold p_c such that $\Pr(\tilde{d}_t = \mathbf{1}_{\{p_t \leq p_c, \theta_t = H\}} | \theta_0) = 1$ for all $t \geq 0$. Moreover, the restriction to strategies in which the low type never certifies is without loss of generality as in equilibrium the low type would never find optimal to certify low quality. □

Proof Proposition 1

Proof. We need to analyze several cases depending on the cost of certification and whether we have investment on equilibrium or not. In absence of investment we have that quality starts at $\theta = H$, it depreciates at a rate λ and $\theta = L$ is an absorbing state. The first set of results characterizes the value function when this is the case.

Equilibria with No Investment

In absence of investment, the only decision for the firm is when to disclose. If the value function is increasing in beliefs, then the certification strategy is characterized by a certification threshold p_c. Let τ be the first time beliefs reach the certification threshold p_c. Direct computation yields the value function which is given by

$$V_L(p_t) = \int_t^\tau e^{-r(s-t)} p_s ds \tag{16}$$

$$V_H(p_t) = \int_t^\tau e^{-r(s-t)} p_s ds + e^{-(r+\lambda)(\tau-t)} \Big(V_H(p_0) - c \Big). \tag{17}$$

The certification threshold p_c is an equilibrium if and only if $V_H(p) \geq V_H(p_0) - c$ for all $p \geq p_c$ so the firm does not want to accelerate certification, and $V_H(p_c) \geq c$ so the firm's benefit of certification is higher than the cost.

Step 1: $V_H(p_c) \geq c$. Using (17) and $p_t = e^{-\lambda t} p_0 = e^{-\lambda t}$ we get

$$\begin{aligned} V_H(p_0) &= \frac{\int_0^\tau e^{-rs} p_s ds}{1 - e^{-(r+\lambda)\tau}} - \frac{e^{-(r+\lambda)\tau}}{1 - e^{-(r+\lambda)\tau}} c \\ &= \frac{1}{r+\lambda} - \frac{e^{-(r+\lambda)\tau}}{1 - e^{-(r+\lambda)\tau}} c, \end{aligned}$$

which is a increasing function τ and so a decreasing function of p_c (τ is decreasing in the threshold). Moreover, $V_H(p_0) \to -\infty$ as $\tau \to 0$; hence, there is a threshold p_c^+ such that

$V_H(p_0) = c$. This means that p_c can be an equilibrium certification threshold only if $p_c \leq p_c^+$. Moreover, $p_c^+ > 0$ if and only if $c < \frac{1}{r+\lambda}$; otherwise, the unique equilibrium has no certification.

Step 2: $V_H(p) \geq V_H(p_0) - c$ **for all** $p \geq p_c$**.** A necessary condition for this to be the case is that $V_H'(p_c) \geq 0$; otherwise, there is ϵ such that $V_H(p_c+\epsilon) < V_H(p_0) - c$. If we differentiate (17) with respect to time we get

$$\begin{aligned}\frac{d}{dt}V_H(p_t) &= -p_t + r\int_t^\tau e^{-r(s-t)}p_s ds + (r+\lambda)e^{-(r+\lambda)(\tau-t)}\Big(V_H(p_0) - c\Big)\\ &= -p_t + r\int_t^\tau e^{-(r+\lambda)(s-t)}p_t ds + (r+\lambda)e^{-(r+\lambda)(\tau-t)}\left(\frac{1}{r+\lambda} - \frac{c}{1-e^{-(r+\lambda)\tau}}\right)\\ &= e^{-(r+\lambda)(\tau-t)}\left(1 - \frac{r}{r+\lambda}p_t\right) - \frac{\lambda}{r+\lambda}p_t - \frac{c(r+\lambda)e^{-(r+\lambda)(\tau-t)}}{1-e^{-(r+\lambda)\tau}}.\end{aligned}$$

Because p_t is decreasing in t we have that $V_H'(p_t) \geq 0$ if and only if $\frac{d}{dt}V_H(p_t) \leq 0$. This is true at time τ iff

$$\left.\frac{d}{dt}V_H(p_t)\right|_{t=\tau} = 1 - p_\tau - \frac{c(r+\lambda)}{1-e^{-(r+\lambda)\tau}} \leq 0.$$

Using $p_\tau = p_c$ and $\tau = -\log(p_c)/\lambda$ we get the condition

$$1 - p_c - \frac{c(r+\lambda)}{1-p_c^{\frac{r+\lambda}{\lambda}}} \leq 0 \tag{18}$$

The left hand side of equation (18) is decreasing in p_c. Hence, there is p_c^- such that (18) holds with equality if and only if $c \geq 1/(r+\lambda)$. Moreover, if this is the case, then condition (18) holds for any $p_c \geq p_c^-$. Hence, p_c^- is a lower bound for the certification threshold.

This is only a necessary conditions, we still have to verify that $V_H(p) \geq V_H(p_0) - c$ for $p > p_c$. Taking the second derivative of $V_H(p_t)$ we get

$$\begin{aligned}\frac{d^2}{dt^2}V_H(p_t) &= (r+\lambda)e^{-(r+\lambda)(\tau-t)}\left(1 - \frac{r}{r+\lambda}p_t - \frac{c(r+\lambda)}{1-e^{-(r+\lambda)\tau}}\right)\\ &\quad - \left(e^{-(r+\lambda)(\tau-t)}\frac{r}{r+\lambda} + \frac{\lambda}{r+\lambda}\right)\dot{p}_t\\ &= (r+\lambda)\left(\frac{d}{dt}V_H(p_t) + \frac{\lambda}{r+\lambda}p_t\right) - \left(e^{-(r+\lambda)(\tau-t)}\frac{r}{r+\lambda} + \frac{\lambda}{r+\lambda}\right)\dot{p}_t\end{aligned}$$

Hence, we have that $\frac{d}{dt}V_H(p_t) = 0$ implies $\frac{d^2}{dt^2}V_H(p_t) > 0$. This means that if at time τ we have $\frac{d}{dt}V_H(p_t) \leq 0$ then it must be true that $\frac{d}{dt}V_H(p_t) \leq 0$ for all $t < \tau$. Thus, we have that

$$V_H(p_\tau) - V_H(p_t) = \int_t^\tau \frac{d}{dt} V_H(p_s) ds \leq 0,$$

so $V_H(p_t) \geq V_H(p_\tau) = V_H(p_0) - c$. The final step is to see in which situations the equilibrium has no investment.

Step 3: Investment Incentives

We can compute the incentives to invest using equation (16) and (17)

$$D(p_t) = e^{-(r+\lambda)(\tau-t)}\Big(V_H(p_0) - c\Big) = e^{-(r+\lambda)(\tau-t)}\left(\frac{1}{r+\lambda} - \frac{c}{1-e^{-(r+\lambda)\tau}}\right).$$

Hence, $D(p_t) < \frac{k}{\lambda}$ for all $t \leq \tau$ if and only if

$$\frac{1}{r+\lambda} - \frac{c}{1-e^{-(r+\lambda)\tau}} < \frac{k}{\lambda}.$$

This conditions is true for any τ if and only if $\frac{1}{r+\lambda} - c < \frac{k}{\lambda}$. Otherwise, this is true if and only if

$$\tau < -\frac{1}{r+\lambda} \log\left(1 - \frac{c}{\frac{1}{r+\lambda} - \frac{k}{\lambda}}\right),$$

which corresponds to the certification time τ consistent with the threshold p_c in the first part of Proposition 1.

Equilibria with Investment

We have already characterized the equilibria that have no investment. The final step is to look at those equilibria in which there is positive investment. The boundary conditions at p_c are given by

$$\begin{aligned} V_H(p_c) = V_H(0) &= V_H(1) - c \\ V_L(p_c) = V_L(0) &= \frac{\lambda a(V_H(1) - c) - ak}{r + \lambda a} \end{aligned} \tag{19}$$

Equation (19) can be rewritten

$$V_H(0) = \left(\frac{r}{\lambda a} + 1\right) V_L(0) + \frac{k}{\lambda},$$

hence

$$D(0) = \frac{rV_L(0)}{\lambda a} + \frac{k}{\lambda}.$$

On the other hand, $t \to D(p_t)$ is a continuous function so in equilibrium we must have that

$$D\left(p_c\right) = D(0) = \frac{k}{\lambda}.$$

Otherwise, the firm would invest when beliefs are just above p_c. We can thus conclude that

$$V_L(0) = V_L(p_c) = 0.$$

This in turn implies that

$$V_H\left(1\right) = \frac{k}{\lambda} + c.$$

Let $\tau = \inf\{p_t : p_\tau = p_c\}$. In equilibrium, $a(p_t) = 0$ implies that for $p_t > p_c$ we have

$$\tau = -\frac{\log p_c}{\lambda}.$$

The value function for the high type is given by

$$V_H(p_t) = \int_t^\tau e^{-r(s-t)} p_s + e^{-(r+\lambda)(\tau-t)} \big(V_H(1) - c\big).$$

Using $p_s = e^{-\lambda(s-t)} p_t$ and $V_H(1) - c = k/\lambda$ we get

$$V_H(p_t) = \frac{p_t}{r+\lambda}\left[1 - \left(\frac{p_c}{p_t}\right)^{\frac{r+\lambda}{\lambda}}\right] + \frac{k}{\lambda}\left(\frac{p_c}{p_t}\right)^{\frac{r+\lambda}{\lambda}}.$$

Similarly,

$$V_L(p_t) = \frac{p_t}{r+\lambda}\left[1 - \left(\frac{p_c}{p_t}\right)^{\frac{r+\lambda}{\lambda}}\right].$$

Now, we can compute p_c using the condition $V_H(1) = c + k/\lambda$ which gives us

$$\left(\frac{1}{r+\lambda} - \frac{k}{\lambda}\right)\left[1 - p_c^{\frac{r+\lambda}{\lambda}}\right] = c,$$

so

$$p_c = \left[1 - \frac{c}{\frac{1}{r+\lambda} - \frac{k}{\lambda}}\right]^{\frac{\lambda}{r+\lambda}}.$$

Intuitively, p_c decreases in c and k. An equilibrium with certification and investment exists iff

$$\frac{1}{r+\lambda} - \frac{k}{\lambda} > c$$

Finally, no certification and no investment is an equilibrium if and only if

$$V_H^{nc}(0) > V_H^{nc}(1) - c,$$

which means that

$$c > \frac{1}{r+\lambda}.$$

□

Proof of Proposition 2

Proof. First, we construct the equilibrium. Using equation (10), together with the optimality condition $\bar{D}(\tau_a) = k/\lambda$ we get

$$\begin{aligned} U_H(0) &= \int_t^{\tau_a} e^{-r(s-t)} p_s ds + e^{-r(\tau_a - t)} \left[U_H(\tau_a) - \bar{D}(\tau_a)\right] + e^{-(r+\lambda)(\tau_a - t)} \frac{k}{\lambda} \\ &= \int_t^{\tau_a} e^{-r(s-t)} p_s ds + e^{-r(\tau_a - t)} U_H(\tau_a) + \left(e^{-(r+\lambda)(\tau_a - t)} - e^{-r(\tau_a - t)}\right) \frac{k}{\lambda}. \end{aligned}$$

Replacing $U_H(\tau_a)$ and evaluation at $t = t_0$ we get

$$U_H(0) = \int_0^{\tau_a} e^{-rs} p_s ds + \int_{\tau_a}^{\tau} e^{-rs}(p_s - k) ds + e^{-r\tau}(U_H(0) - c) + \left(e^{-(r+\lambda)\tau_a} - e^{-r\tau_a}\right) \frac{k}{\lambda}. \quad (20)$$

Computing the integral of the price in equation (20) yields

$$\begin{aligned} h(\tau, \tau_a) \equiv \int_0^{\tau} e^{-rs} p_s ds &= \frac{1}{r+\lambda} - \frac{e^{-(r+\lambda)\tau_a}}{r+\lambda} + \frac{e^{-r\tau_a} - e^{-r\tau}}{r} \\ &\quad \frac{e^{-(r+\lambda)\tau_a} - e^{-(r+\lambda)\tau}}{r+\lambda} + \frac{e^{-(r+\lambda)\tau + \lambda\tau_a} - e^{-r\tau_a}}{r+\lambda} \\ &= \frac{1}{r+\lambda} + \frac{e^{-r\tau_a} - e^{-r\tau}}{r} - \frac{e^{-(r+\lambda)\tau}}{r+\lambda} + e^{-r\tau_a} \frac{e^{-(r+\lambda)(\tau - \tau_a)} - 1}{r+\lambda} \end{aligned}$$

Replacing in and (20) and defining $v \equiv U_H(0) - c$ we get (12). Computing g as the inverse of (7) we get

$$g(\tau,\tau_a) \equiv \frac{k}{\lambda}\frac{r+\lambda}{r}e^{(r+\lambda)(\tau-\tau_a)} - \frac{k}{r}$$

In equilibrium, if $\tau_a > 0$, it must be the case that

$$g(\tau,\tau_a) = v(\tau,\tau_a).$$

Finally, suppose that $v(\tau,0) \geq g(\tau,0)$. By definition, this means

$$\bar{D}(0) = e^{-(r+\lambda)\tau}\frac{rv+k}{r+\lambda} \geq \frac{k}{\lambda},$$

so $\tau_a = 0$ is optimal for the seller.

The next step is to show that for any $\tau > \tau_c$ an equilibrium exists and $\tau_a < \tau$. By continuity, it suffices to show that $v(\tau,\tau) > g(\tau,\tau)$. First, we have that $g(\tau,\tau)$ is

$$g(\tau,\tau) = \frac{k}{\lambda}\frac{r+\lambda}{r} - \frac{k}{r} = \frac{k}{\lambda}$$

Evaluating the RHS at $\tau_a = \tau$ we get

$$v(\tau,\tau) = \frac{\frac{1}{r+\lambda} - \frac{e^{-(r+\lambda)\tau}}{r+\lambda} + \left(e^{-(r+\lambda)\tau} - e^{-r\tau}\right)\frac{k}{\lambda} - c}{1-e^{-r\tau}}$$

If $c < \frac{1}{r+\lambda} - \frac{k}{\lambda}$ then $\lim_{\tau\to\infty} v(\tau,\tau) > \frac{k}{\lambda}$. Next, evaluating the limit at τ_c we obtain

$$v(\tau_c,\tau_c) = \frac{k}{\lambda}$$

As

$$\frac{d}{d\tau}v(\tau,\tau) = \frac{re^{-r\tau}}{1-e^{-r\tau}}\left(\frac{k}{\lambda} - v(\tau,\tau)\right) + \frac{(r+\lambda)e^{-(r+\lambda)\tau}}{1-e^{-r\tau}}\left(\frac{1}{r+\lambda} - \frac{k}{\lambda}\right),$$

we get that $v(\tau,\tau) = k/\lambda$ implies $\frac{d}{d\tau}v(\tau,\tau) > 0$. Hence, $v(\tau_c,\tau_c) = \frac{k}{\lambda}$ implies that $v(\tau,\tau) > \frac{k}{\lambda}$ for all $\tau > \tau_c$. □

Proof of Proposition 3

Proposition 3(i)

First we note that for any $\tau_a^0, \tau_a^1 \in \mathcal{E}(\tau)$ it is the case that $\tau_a^1 > \tau_a^0$ implies $U(0|\tau, \tau_a^0) > U(0|\tau, \tau_a^1)$. This follows from the fact that for any $\tau_a > 0$

$$U(0|\tau, \tau_a) = g(\tau, \tau_a) + c$$

is decreasing in τ_a and when $\tau_a = 0$ we have $U(0|\tau, 0) > g(\tau, 0) + c$. Hence, it is sufficient to show that there is τ such that $U(0|\tau, \overline{\tau}_a) > V^{nc}(p_0)$. We complete the proof with two lemmas. In Lemma 3 we show that there is a benefit setting $\tau > \tau_c$, while in Lemma 4 we show that $\tau = \infty$ is not optimal.

Lemma 3. *Suppose that* $\frac{1}{r+\lambda} - \frac{k}{\lambda} > c$ *then* $\frac{d}{d\tau} v(\tau, \tau_a(\tau))\big|_{\tau=\tau_c} > 0$.

Proof. Suppose that $\tau_a > 0$. In this case, we have that

$$\begin{aligned} \frac{d}{d\tau} v(\tau, \tau_a(\tau)) &= v_\tau(\tau, \tau_a) + v_{\tau_a}(\tau, \tau_a)\tau_a'(\tau) \\ \tau_a'(\tau) &= -\frac{v_\tau(\tau, \tau_a) - g_\tau(\tau, \tau_a)}{v_{\tau_a}(\tau, \tau_a) - g_{\tau_a}(\tau, \tau_a)} \end{aligned} \tag{21}$$

Where,

$$g_\tau(\tau, \tau_a) = \frac{k}{\lambda}\frac{(r+\lambda)^2}{r} e^{(r+\lambda)(\tau-\tau_a)} = (r+\lambda) g(\tau, \tau_a) + (r+\lambda)\frac{k}{r} \tag{22}$$

$$g_{\tau_a}(\tau, \tau_a) = -\frac{k}{\lambda}\frac{(r+\lambda)^2}{r} e^{(r+\lambda)(\tau-\tau_a)} = -g_\tau(\tau, \tau_a) \tag{23}$$

$$v_\tau(\tau, \tau_a) = \frac{e^{-r\tau}}{1-e^{-r\tau}}\left(\frac{1-k}{r} - v\right) + \frac{e^{-(r+\lambda)\tau} - e^{-r\tau-\lambda(\tau-\tau_a)}}{1-e^{-r\tau}} \tag{24}$$

$$v_{\tau_a}(\tau, \tau_a) = \frac{e^{-r\tau_a}\left(k\left(1-e^{-\lambda\tau_a}\right)(r+\lambda)^2 + \lambda^2\left(e^{-(r+\lambda)(\tau-\tau_a)} - 1\right)\right)}{\lambda(r+\lambda)\left(1-e^{-r\tau}\right)} \tag{25}$$

Evaluating at $\tau = \tau_a = \tau_c$ and using $v(\tau_c, \tau_c) = g(\tau_c, \tau_c) = k/\lambda$ we get

$$\begin{aligned}
g_\tau(\tau_c, \tau_c) &= (r+\lambda)\frac{k}{\lambda} + (r+\lambda)\frac{k}{r} > 0 \\
g_{\tau_a}(\tau_c, \tau_c) &= -(r+\lambda)\frac{k}{\lambda} - (r+\lambda)\frac{k}{r} < 0 \\
v_\tau(\tau_c, \tau_c) &= \frac{(r+\lambda)e^{-r\tau_c}}{1-e^{-r\tau_c}}\left(\frac{p_c}{r+\lambda} - \frac{k}{\lambda}\right) \\
v_{\tau_a}(\tau_c, \tau_c) &= \frac{k}{\lambda}\frac{\left(1-e^{-\lambda\tau_c}\right)(r+\lambda)}{e^{r\tau_c}-1} = \frac{(r+\lambda)e^{-r\tau_c}}{1-e^{-r\tau_c}}\frac{k}{\lambda}(1-p_c) > 0
\end{aligned}$$

Noting that we can write

$$v_\tau(\tau_c, \tau_c) = \frac{(r+\lambda)p_c e^{-r\tau_c}}{1-e^{-r\tau_c}}\left(\frac{1}{r+\lambda} - \frac{k}{\lambda}\right) - v_{\tau_a}(\tau_c, \tau_c).$$

Replacing in the equation for $\frac{d}{d\tau}v(\tau, \tau_a(\tau))$

$$\frac{d}{d\tau}v(\tau, \tau_a(\tau))\big|_{\tau=\tau_c} = \frac{(r+\lambda)p_c e^{-r\tau_c}}{1-e^{-r\tau_c}}\left(\frac{1}{r+\lambda} - \frac{k}{\lambda}\right) + v_{\tau_a}(\tau_c, \tau_c)(\tau_a'(\tau_c) - 1).$$

Moreover, we have

$$\begin{aligned}
\tau_a'(\tau_c) - 1 &= -\frac{v_\tau(\tau_c, \tau_c) + v_{\tau_a}(\tau_c, \tau_c)}{v_{\tau_a}(\tau_c, \tau_c) - g_{\tau_a}(\tau_c, \tau_c)} \\
&= -\frac{\frac{(r+\lambda)p_c e^{-r\tau_c}}{1-e^{-r\tau_c}}\left(\frac{1}{r+\lambda} - \frac{k}{\lambda}\right)}{v_{\tau_a}(\tau_c, \tau_c) - g_{\tau_a}(\tau_c, \tau_c)},
\end{aligned}$$

which means that

$$\frac{d}{d\tau}v(\tau, \tau_a(\tau))\big|_{\tau=\tau_c} = \frac{(r+\lambda)p_c e^{-r\tau_c}}{1-e^{-r\tau_c}}\left(1 - \frac{v_{\tau_a}(\tau_c, \tau_c)}{v_{\tau_a}(\tau_c, \tau_c) - g_{\tau_a}(\tau_c, \tau_c)}\right) > 0$$

Finally, suppose that $\tau_a = 0$. In this case, we have that

$$\frac{d}{d\tau}v(\tau, \tau_a(\tau)) = v_\tau(\tau, \tau_a) = \frac{e^{-r\tau}}{1-e^{-r\tau}}\left(\frac{1-k}{r} - v\right) > 0.$$

Where the last inequality follows from the fact that $(1-k)/r$ is the payoff in the first best and so it is necessarily grater than v. □

Lemma 4. *Suppose that* $\frac{1}{r+\lambda} - \frac{k}{\lambda} > c$ *then there is* τ *such that* $U_H(0|\tau, \overline{\tau}_a) > V_H^{nc}(p_0)$.

Proof. We start showing that $\lim_{\tau\to\infty}\left(\overline{\tau}_a(\tau)-\tau\right)=0$ and $\lim_{\tau\to\infty}\overline{\tau}_a'(\tau)=1$. This implies that

$$\lim_{\tau\to\infty}\frac{d}{d\tau}v(\tau,\overline{\tau}_a)=\lim_{\tau\to\infty}v_\tau(\tau,\overline{\tau}_a)+v_{\tau_a}(\tau,\overline{\tau}_a)\overline{\tau}_a'(\tau)=v_\tau(\tau,\overline{\tau}_a)+v_{\tau_a}(\tau,\overline{\tau}_a)=\lim_{\tau\to\infty}\frac{d}{d\tau}v(\tau,\tau).$$

The final step is to show that $\frac{d}{d\tau}v(\tau,\tau)<0$ for τ arbitrarily large.

Step 1: $\lim_{\tau\to\infty}\left(\overline{\tau}_a(\tau)-\tau\right)=0$ and $\lim_{\tau\to\infty}\overline{\tau}_a'(\tau)=1$.

Given that $v(\tau,\tau_a)$ is bounded above and $g(\tau,0)\to\infty$ as $\tau\to\infty$ it must be the case that $\overline{\tau}_a>0$ for τ sufficiently large. Defining $x\equiv\exp(-r\tau)$ and $y\equiv\exp(-r\tau_a)$ we can write the equilibrium condition for τ_a in terms of x and y as

$$\frac{k}{\lambda}\frac{r+\lambda}{r}y^{1+\frac{\lambda}{r}}=x^{1+\frac{\lambda}{r}}\left[\frac{k}{r}+\frac{\left(\frac{1-x^{1+\frac{\lambda}{r}}}{r+\lambda}+\frac{y-x}{r}+y^r\frac{\left(\frac{x}{y}\right)^{1+\frac{\lambda}{r}}-1}{r+\lambda}\right)-\frac{(y-x)k}{r}+\frac{\left(y^{1+\frac{\lambda}{r}}-y\right)k}{\lambda}-c}{1-x}\right].$$

By direct inspection of the above equation we conclude that the limit when $\tau\to\infty$ which corresponds to the limit when $x\to 0$ is given by

$$\lim_{x\to 0}(y-x)=0.$$

Moreover, replacing x and y into equation (21)

$$\overline{\tau}_a'=-\frac{\frac{x}{1-x}\left(\frac{1-k}{r}-v\right)+\frac{x^{1+\frac{\lambda}{r}}-\left(\frac{x}{y}\right)^{1+\frac{\lambda}{r}}y}{(1-x)}-\frac{k}{\lambda}\frac{(r+\lambda)^2}{r}\left(\frac{y}{x}\right)^{1+\frac{\lambda}{r}}}{\frac{y\left(k\left(1-y^{\frac{\lambda}{r}}\right)(r+\lambda)^2+\lambda^2\left(\left(\frac{x}{y}\right)^{1+\frac{\lambda}{r}}-1\right)\right)}{\lambda(r+\lambda)(1-x)}+\frac{k}{\lambda}\frac{(r+\lambda)^2}{r}\left(\frac{y}{x}\right)^{(1+\frac{\lambda}{r})}}.$$

and taking the limit when $x\to 0$ we get

$$\lim_{x\to 0}\overline{\tau}_a'=-\frac{-\frac{k}{\lambda}\frac{(r+\lambda)^2}{r}}{-\frac{k}{\lambda}\frac{(r+\lambda)^2}{r}}=1.$$

Step 2: We show that $\lim_{x\to 0}\frac{dv(\tau(x),\tau(x))}{dx}>0$. This, along with Step 1, implies that the optimal τ is interior. Substituting $x = e^{-r\tau}$ into $v(\tau, \tau)$ yields

$$v(\tau(x), \tau(x)) = \frac{\frac{1}{r+\lambda} - \frac{x^{1+\frac{\lambda}{r}}}{r+\lambda} + \left(x^{1+\frac{\lambda}{r}} - x\right)\frac{k}{\lambda} - c}{1-x}.$$

Differentiating $v(\tau(x), \tau(x))$

$$\frac{dv(\tau(x), \tau(x))}{dx} = \frac{-\frac{x^{\frac{\lambda}{r}}}{r} + \left(\frac{r+\lambda}{r}x^{\frac{\lambda}{r}} - 1\right)\frac{k}{\lambda}}{1-x} + \frac{\frac{1}{r+\lambda} - \frac{x^{1+\frac{\lambda}{r}}}{r+\lambda} + \left(x^{1+\frac{\lambda}{r}} - x\right)\frac{k}{\lambda} - c}{(1-x)^2},$$

and evaluating at $x = 0$ we get

$$\left.\frac{dv(\tau(x), \tau(x))}{dx}\right|_{x=0} = \frac{1}{r+\lambda} - \frac{k}{\lambda} - c > 0.$$

By Step 1 we have that

$$\lim_{x\to 0}\frac{dv(\tau(x), \tau(x))}{dx} = \lim_{x\to 0}\frac{d}{dx}v(\tau(x), \overline{\tau}_a(\tau(x))).$$

Hence, we have that for τ arbitrarily large

$$\frac{dv(\tau, \overline{\tau}_a(\tau))}{d\tau} = \frac{dv(\tau(x), \overline{\tau}_a(\tau(x)))}{dx}\frac{dx}{d\tau} = -re^{-r\tau}\frac{dv(\tau(x), \overline{\tau}_a(\tau(x)))}{dx} < 0$$

so the τ that maximizes $v(\tau, \overline{\tau}_a(\tau))$ is interior. □

Proposition 3(ii)

Let's define $f(t_a, c) := g(\tau, t_a) - v(\tau, t_a)$. The derivative of $f(\cdot, c)$ with respect to c is $\frac{1}{1-e^{-r\tau}} > 0$ so we have that $f(t_a, c_1) \geq f(t_a, c_0)$ for any $c_1 > c_0$. Accordingly, Lemma 1 in Milgrom and Roberts (1994) implies that $\underline{\tau}_a(c_1) = \inf\{t_a \in [0, \tau] : f(t_a, c_1) \leq 0\} \geq \inf\{t_a \in [0, \tau] : f(t_a, c_0) \leq 0\} = \underline{\tau}_a(c_0)$ and $\overline{\tau}_a(c_1) = \sup\{t_a \in [0, \tau] : f(t_a, c_1) \geq 0\} \geq \sup\{t_a \in [0, \tau] : f(t_a, c_0) \geq 0\} = \overline{\tau}_a(c_0)$.

Proposition 3(iii)

We prove the result only for $\sup_{\tau\geq\tau_c}\underline{U}_H(0|\tau)$ as the proof for $\sup_{\tau\geq\tau_c}\overline{U}_H(0|\tau)$ is analogous. Let's consider $c_1 > c_0$; from Proposition 3(ii) we have that $\underline{\tau}_a(c_1) \geq \underline{\tau}_a(c_0)$; which means that it suffices to show that $U_H(1|\tau, \tau_a, c)$ is decreasing in τ_a. Fix τ and consider the case with $\underline{\tau}_a(c_0) > 0$. Using equation (13), we have that

$$U_H(0|\tau, \overline{\tau}_a(\tau, c), c) - c = g(\tau, \overline{\tau}_a(\tau, c)). \tag{26}$$

Hence, the firm's ex-ante profit given an investment threshold t_a is

$$\begin{aligned} U_H(0|\tau, t_a, c) &= h(\tau, t_a) - (e^{-rt_a} - e^{-r\tau})\frac{k}{r} - e^{-r\tau}(1 - p_\tau)\frac{k}{r+\lambda} \qquad (27) \\ &+ e^{-r\tau}\left(\frac{\lambda}{r+\lambda} + \frac{r}{r+\lambda}p_\tau\right)\big(U_H(0|\tau, t_a, c_0) - c_0\big) \\ &= h(\tau, t_a) - (e^{-rt_a} - e^{-r\tau})\frac{k}{r} - e^{-r\tau}(1 - p_\tau)\frac{k}{r+\lambda} \qquad (28) \\ &+ e^{-r\tau}\left(\frac{\lambda}{r+\lambda} + \frac{r}{r+\lambda}p_\tau\right) g(\tau, t_a), \end{aligned}$$

where the function h is defined in Proposition 3 and in the second equation we have replaced equation (26). The derivative with respect to t_a is

$$\frac{e^{-(r+\lambda)t_a}\left(r\left(kr(r+\lambda) - \lambda^2\right)e^{\lambda t_a} - k(r+\lambda)^3 e^{\lambda\tau} - kr(\lambda+r)^2 + \lambda^2 r e^{-(r+\lambda)(\tau - t_a)}\right)}{\lambda r(\lambda + r)}$$

The sign of the previous expression is determined by the sign of the denominator which is

$$-k(r+\lambda)\big((r+\lambda)^2 e^{\lambda\tau} - r^2 e^{\lambda t_a} + r(\lambda + r)\big) - \lambda^2 r\big(e^{\lambda t_a} - e^{-(r+\lambda)(\tau - t_a)}\big) < 0.$$

Hence, we have that $\frac{\partial}{\partial t_a}U_H(0|\tau, t_a, c) < 0$. Next, we consider the case with $\tau_a(c_0) = 0$. If $\tau_a(c_1) > 0$ then $U_H(0|\tau, \tau_a(c_0), c_0)$ is strictly greater than (28) so the previous argument for the case with $\tau_a(c_0)$ applies and $U_H(0|\tau, \tau_a(c_0), c_0) > U_H(0|\tau, \tau_a(c_1), c_1)$. Finally, the case with $\tau_a(c_0) = \tau_a(c_1) = 0$ is trivial as both policies have the same investment, the same certification and one has a lower cost.

Proof of Lemma 2

Proof. Let $0 < S_1 < S_2 < \cdots$ be the certification times generated by the certification strategy d. Given this certification strategy, the process $\{S_n\}$ is a renewal process and $\{(\theta_t, R_t)\}$ is a regenerative process with regeneration points S_n. Noting that $\{S_n\}$ has a non-lattice distribution, it follows from the renewal theorem (Asmussen, 2003, Theorem 1.2,

p. 170) that

$$\mathbb{P}^e(R=1) = \frac{E\left[\int_0^{S_1} \mathbf{1}_{R_t=1} dt\right]}{E[S_1]} = \frac{\tau}{E[S_1]} \tag{29}$$

and

$$\mathbb{P}^e(R=1,\theta=H) = \frac{E\left[\int_0^{S_1} \mathbf{1}_{\theta_t=H}\mathbf{1}_{R_t=1} dt\right]}{E[S_1]} = \frac{\int_0^{\tau} E[\mathbf{1}_{\theta_t=H}]dt}{E[S_1]}, \tag{30}$$

where the second equality in (30) uses the fact that $\mathbf{1}_{\theta_t=H}\mathbf{1}_{R_t=1} = 0$ for $t \in (\tau, S_1)$ and $\mathbf{1}_{\theta_t=H}\mathbf{1}_{R_t=1} = \mathbf{1}_{\theta_t=1}$ for $t \in [0,\tau]$. Combining (29) and (30) we get

$$\mathbb{P}^e(\theta = H|R=1) = \frac{\int_0^{\tau} E[\mathbf{1}_{\theta_t=1}]dt}{\tau} = \frac{1}{\tau}\int_0^{\tau} p_t dt$$

□

Proof of Proposition 5(i)

Proof. The monotonicity of $\underline{\tau}_a^u$ and $\overline{\tau}_a^u$ follows the proof in Proposition 3(ii) using the fact that for fixed τ_a, τ we have $\partial v^u/\partial c < 0$. The monotonicity of the expected payoff follows the proof in Proposition 3(iii). Whenever the equilibrium τ_a is interior we have

$$\begin{aligned} U_H^u(0|\tau,\tau_a,c) = {} & \frac{p_e}{r}\left(1-e^{-r\tau}\right) - (e^{-r\tau_a} - e^{-r\tau})\frac{k}{r} - e^{-r\tau}(1-p_\tau)\frac{k}{r+\lambda} \\ & + e^{-r\tau}\left(\frac{\lambda}{r+\lambda} + \frac{r}{r+\lambda}p_\tau\right) g(\tau,\tau_a). \end{aligned}$$

After some algebra, we find that the sign of the derivative is determined by

$$\begin{aligned} \frac{\partial}{\partial \tau_a} U_H^u(0|\tau,\tau_a,c) \propto {} & -k\tau e^{-(r+\lambda)\tau_a}\left(r(r+\lambda) + (r+\lambda)^2 e^{\lambda\tau} - r^2 e^{\lambda\tau_a}\right) \\ & - \lambda\left(1-e^{-\lambda(\tau-\tau_a)}\right)\left(1-e^{-r\tau}\right) < 0 \end{aligned}$$

Hence, given that for any $c_1 > c_0$ we have that $\underline{\tau}_a^u(c_1) > \underline{\tau}_a^u(c_1)$ and $\overline{\tau}_a^u(c_1) > \overline{\tau}_a^u(c_1)$ we can conclude that $U_H^u(0|\tau,\underline{\tau}_a^u(c_1),c_1) < U_H^u(0|\tau,\underline{\tau}_a^u(c_0),c_0)$ and $U_H^u(0|\tau,\overline{\tau}_a^u(c_1),c_1) < U_H^u(0|\tau,\overline{\tau}_a^u(c_0),c_0)$. The case in which $\tau_a = 0$ is trivial as both policies have the same investment, the same certification and one has a lower cost. □

Proof of Proposition 5(ii)

Proof. The first step is to verify that for fixed τ, τ_a we have $v^u(\tau, \tau_a) < v(\tau, \tau_a)$. Noting that the only term that differs is the one regarding the present value of prices, it suffices to show that

$$\frac{\int_0^\tau e^{-rt} p_t dt}{1 - e^{-r\tau}} > \frac{p_e}{r} = \frac{\int_0^\tau p_t dt}{r\tau}.$$

Let

$$\Delta(r) \equiv \frac{\int_0^\tau re^{-rt} p_t dt}{1 - e^{-r\tau}} - \frac{\int_0^\tau p_t dt}{\tau}.$$

Using l'Hôpital rule we find that $\lim_{r \to 0} \Delta(r) = 0$. Differentiating with respect to r we get

$$\Delta'(r) = \frac{e^{r\tau} - 1 - r\tau}{r(\lambda + r)\left(e^{r\tau} - 1\right)^2} \Big[re^{r\tau} + \lambda e^{r(\tau - \tau_a)} - (\lambda + r) + r \Big(e^{-\lambda(\tau - \tau_a)} - e^{-\lambda\tau} \Big) \Big] > 0,$$

where we have used the inequality $e^{rt} \geq 1 + rt$. Thus, we have that $\Delta(r) > 0$ for all $r > 0$, which means that $v^u(\tau, \tau_a) - g(\tau, \tau_a) < v(\tau, \tau_a) - g(\tau, \tau_a)$. Lemma 1 in Milgrom and Roberts (1994) implies that $\underline{\tau}_a = \inf\{t_a \in [0, \tau] : v(\tau, \tau_a) - g(\tau, \tau_a) \leq 0\} \leq \inf\{t_a \in [0, \tau] : v^u(\tau, \tau_a) - g(\tau, \tau_a) \leq 0\} = \underline{\tau}_a^u$ and $\overline{\tau}_a = \sup\{t_a \in [0, \tau] : v(\tau, \tau_a) - g(\tau, \tau_a) \geq 0\} \leq \sup\{t_a \in [0, \tau] : v^u(\tau, \tau_a) - g(\tau, \tau_a) \geq 0\} = \overline{\tau}_a^u$. □

Next we compare the ex-ante payoffs with and without observable expiration. We consider two cases, $\underline{\tau}_0 > 0$ and $\underline{\tau}_a = 0$. Let's consider the case $\underline{\tau}_a > 0$. From part (ii) we can conclude that $\underline{\tau}_a^u \geq \underline{\tau}_a > 0$. Hence,

$$v^u(\tau, \underline{\tau}_a^u) = g(\tau, \underline{\tau}_a^u) \leq g(\tau, \underline{\tau}_a) = v(\tau, \underline{\tau}_a)$$

so $\underline{U}_H^u(0) = v^u(\tau, \underline{\tau}_a^u) + c \leq v(\tau, \underline{\tau}_a) + c = \underline{U}_H(0)$. Next, we consider the case $\underline{\tau}_a = 0$. If $\underline{\tau}_a^u = 0$ then the equilibrium with and without unobservable expiration are equivalent so there is nothing to prove. If $\underline{\tau}_a^u > 0$ then we have

$$v^u(\tau, \underline{\tau}_a^u) = g(\tau, \underline{\tau}_a^u) \leq g(\tau, \underline{\tau}_a) \leq v(\tau, \underline{\tau}_a),$$

which yields the desired conclusion. The proof for the payoffs in the worst equilibrium, $\{\overline{U}_H^u(0), \overline{U}_H(0)\}$, is analogous.

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