Dynamic Certification, Reputation for Quality and Industry Standard*

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Abstract

We study firm's incentives to build and maintain reputation for quality, when quality is persistent and can be certified at a cost. We characterize all Markov-perfect equilibria where the timing of certification and investment depend only on firm's reputation. They vary in frequency of certification and payoffs, including low payoffs due to over-certification trap. We contrast the MPEs with the highest-payoff equilibria. We interpret that industry certification standards can help firms coordinate on such good equilibria. The optimal standard allows firms to maintain high quality forever, once it is reached for the first time, and it can be either lenient or harsh - endowing firms with multiple or one chance to improve and certify quality.

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1 Introduction

Firms can affect the quality of their products by investing in physical or human capital, research and development, or organizational design. Customers often do not directly observe

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these investments or their results, giving rise to a moral hazard problem that leads to the under-provision of quality. That problem can be mitigated if the firm can invest to build a reputation for quality. However, for the reputation to be credible, customers need to observe signals of quality. These are often provided by the firm via voluntary, costly disclosures. To be credible, such disclosures often are certified by a third party. Examples range from health care (for example, accreditation of HMOs by NCQA, described below), child care (for example, accreditation provided by the National Association for the Education of Young Children), and supplier relationships in B2B contracting (for example, ISO 9000 certification with over one million organizations independently certified worldwide).[1]

In this paper we study the role that an industry standard for voluntary certification plays in mitigating the under provision of quality and in avoiding over-certification trap. Such self-regulation by incumbents has been criticized as a way to increase barriers to entry (see for example Lott (1987)). We ask if it can also be efficiency-enhancing by allowing firms to coordinate on equilibria that provide better incentives to invest in quality and stronger reputations at a lower cost of certification. To this end, we analyze two types of equilibria. The first class is Markov-Perfect equilibria in which firm's certification and investment strategies depend only on current reputation, which we define as the market belief about current quality. We interpret these equilibria as plausible outcomes in case the industry does not self-regulate. The second class we study are optimal perfect Bayesian equilibria, in which the market expectation of firms' certification (and investment) strategy can be a function of the whole history of the game and not just current reputation. For example, industry regulation can prevent firms from re-certifying too soon since the last successful or failed attempt to certify.

We adopt a capital-theoretic approach to modeling both quality and reputation, as in Board and Meyer-ter-Vehn (2013). The firm continuously and privately chooses quality investment. Quality is persistent, changing stochastically between two states, high and low, with the transition rates depending on the instantaneous investment flows, so that current quality reflects all past investments. Reputation drifts up if the firm is believed to be investing and drifts down if not. Profit flows depend on firm's reputation, which is defined as market's belief about its quality.[2] This setting seems realistic for many markets. For

[1]Other sources of information about product quality include mandatory disclosure (such as nutritional facts), third-party initiated reviews (such as reviews on Cnet.com), and consumer reports (word of mouth or consumer reports on Amazon.com). See a survey by Dranove and Jin (2010).

[2]Profits can increase in perceived quality either because good reputation leads to a bigger demand for the product or because it allows the firm to charge a higher price, or both. For empirical evidence that

example, in the health-care industry, HMOs invest in processes and personnel to provide high-quality services, quality is persistent since human capital and organizational capital are persistent but maintaining quality requires continuous investment to attract and retain talent, and to react to changes in medical practice or technology. Moreover, quality is hard to observe by individual customers and an important source of information is the National Committee of Quality Assurance (NCQA) that since 1991 offers HMOs voluntary certification program. The certificates expire in three years and total costs (direct fees and indirect costs) of preparing accreditation range from $\$30,000$ to $\$100,000$ depending on the size of an HMO (and other characteristics; see Jin (2005) for a detailed description of the NCQA program).

Quality is known privately by the firm but at any time it can be credibly revealed/certified to the market. We model certification as a costly disclosure that allows the firm to credibly and perfectly convey its current (and somewhat persistent) quality to the market. This is similar to certification in Jovanovic (1982) and Verrecchia (1983), with the main differences being that in our model quality is endogenous and disclosure is dynamic rather than static. Though we do not model the source of this disclosure cost, we interpret it as representing the fee charged by a certifier in exchange for its certification and dissemination services (in the spirit of Lizzeri (1999)), plus any costs necessary to allow the certifier verify the firm's quality.

Since the firm is privately informed about its quality, the market learns about quality not only from certification but also from the failure to apply for accreditation. This leads to multiplicity of equilibria that differ in terms of the frequency of certification. The difference in the two classes of equilibria we study is how these market expectations change in response to history. In the Markov-perfect equilibria market expectations are stationary - they depend only on the current reputation. In the optimal equilibria, the expected frequency of future certification can depend on past behavior. For example, if a high quality firm fails to maintain quality and re-certify, the market can expect a more frequent certification and less investment in the future.

There are two sets of results in the paper. First, we characterize Markov-perfect equilibria. When certification costs are low, there is a range of equilibria with different frequencies of certification. In particular, there exist equilibria with a high frequency of certification in which all the benefits of reputation for the high quality firms are dissipated by excessive

certification increases demand, see for example Xiao (2007) in the context of voluntary accreditation of child care centers, and other examples in Dranove and Jin (2010).

certification, an effect we call an over-certification trap. Moreover, we show that under our assumptions the Markov-perfect equilibria do not create any value for firms that start at low quality. That is, even though in some Markov-perfect equilibria the firm invests in quality and eventually manages to certify it, all MPEs, for all positive costs of certification, result in the same payoff to the low-quality firm, as if quality could never be improved.[3] Moreover, in MPEs with on-path investment in quality, quality is impermanent: even though the firm has the technology to maintain quality forever, on path expected quality slowly drops after certification.

The second set of results characterizes the best equilibria. The best equilibrium not only delivers higher payoffs than any MPE, but also differs qualitatively from all MPEs. For low certification costs we show that in the best equilibrium the ex-ante payoff of the low-quality firm is strictly higher and increases as cost of certification goes down, converging to the first-best payoff when the cost of certification declines to zero. Moreover, once the firm reaches high quality, it is maintained forever on the equilibrium path in contrast with all MPEs.

In summary, the analysis implies that an industry standard for voluntary certification could allow firms to create and reap benefits from building and maintaining reputation and avoid the over-certification trap. An important feature of such a system is that it keeps track of the time since last certification and sets the duration (i.e. the time the high quality firm is expected to re-certify) optimally: a short duration creates too much costs of certification that by reducing the value of reputation reduce incentives to invest; a long duration makes just-certified firms rest on their laurels and shirk since today's investments have small effect on long-term quality. Finally, the optimal equilibrium can be implemented by a system that keeps track of the time since last certification and a binary indicator whether the firm is still in the system or not (a punishment can be implemented by removing the firm from the industry certification program and letting it to its own devices).

Our somewhat paradoxical result about the role of certification in MPEs stresses that certification can be a double-edged sword: on one hand it allows firms to reap benefits of investments in quality, on the other hand, it can create an (over) certification trap, if the market expects the firm to re-certify frequently. Paradoxically, high-quality firms caught in such a trap earn lower profits than if no certification were possible - this happens even in the MPE with the most investment in quality. The intuition for the low payoffs in any MPE

[3]This stark result depends on the assumption that if the firm invests maximally quality never drops. However, as we discuss later, the intuition for over-certification trap and the corresponding benefit of coordination on better equilibria is robust.

is as follows. First, if certification takes place only after beliefs drop below some level, the firm cannot be investing in quality above that threshold since otherwise market beliefs would never reach it (recall that in our model, expected quality improves when the firm invests and deteriorates if it does not). Hence, it is not possible to forever maintain high quality in any MPE and payoffs of a high-quality firm are bounded away from first-best. Second, the firm with the lowest reputation cannot have strict incentives to invest in quality either. If it did, the firm would also have strict incentives to invest before it fails to certify and market beliefs would never reach the certification threshold. As cost of certification gets lower, the firm in equilibrium certifies more and more often and all the savings are dissipated by excessively frequent certification.

It may be at first counter-intuitive that less-frequent certification improves incentives to invest in quality. The intuition is that with less-frequent certification, the total expected continuation profits from certifying high quality are higher since less resources are spent on certifying. Moreover, there is a positive feedback effect: higher payoffs from high quality increase incentives for investment, and that increases payoffs even further and so on.

The optimal equilibrium takes one of two forms, *harsh* or *lenient*. The difference between them is what happens when the firm starts at low quality. In the harsh equilibrium, the low quality firm has to wait a long time till certification, so it passes it with a high probability, but failure is harshly punished (the punishment can be interpreted as the firm being excluded from the industry certification program while maintaining the option to certify independently according to one of the MPEs we described first). In the lenient equilibrium, the firm gets a shorter time to first certification, but failure is not punished (beyond the reset of reputation to the lowest level) – the equilibrium simply restarts. In other words, the firm is given multiple chances to improve and certify its quality no matter how many times it has failed before. Intuitively, the harsh equilibrium provides stronger incentives and hence can economize on certification costs, but it also sometimes triggers inefficient punishment on the equilibrium path (false-positive when the firm is unlucky in achieving high quality by the deadline despite appropriate investment). If certification costs are small, the best equilibrium is lenient. On the other hand, if certification costs are large and quality improves sufficiently easily (both in terms of cost of investment and arrival rate of improvements), the optimal equilibrium is harsh.

While we propose to interpret the difference between the MPEs and the best equilibria as a potential benefit of an industry standard, in practice firms can affect market expectations about the frequency of certification (and hence try to coordinate on better equilibria) in other ways too. For example, they sometimes resort to third parties to create certification with a pre-announced duration.[4] Therefore, our analysis can be interpreted more broadly as showing in an equilibrium setup first the potential costs of over-certification, and second the benefits of managing market expectations about timing of certification.

1.1 Related Literature

As we mentioned above, our paper can be viewed as a dynamic version of Jovanovic (1982); Verrecchia (1983) with endogenous quality. Our model of quality and interpretation of reputation is as in Board and Meyer-ter Vehn (2013).[5] Similar papers that consider incentives to invest in quality with exogenous public news include Dilme (2016); Halac and Prat (2016). There are two main differences between our paper and this literature. First is how we model information: in our model it is generated endogenously by the firm, while in their models the market observes exogenous signals about the quality. Second, these previous models study only Markov-Perfect equilibria, and our model contrasts MPEs with the optimal equilibria. The contrast between what can be achieved in each class is the main result of our paper. An implication of these results (that we do not to emphasize) is that focusing on MPEs in reputation models can rule out realistic behavior.[6].

A strand of the literature studies certification, focusing on the behavior of a monopoly certifier who can commit in advance to both a certification fee and a disclosure rule (see e.g., Lizzeri (1999), Albano and Lizzeri (2001)). In this paper we take the certification technology as exogenous and focus instead on firm's investment behavior, but we believe our model could be also used to study profit-maximizing certifiers. Our model suggests that an optimal strategy of a certifier would involve a non-trivial decision about price as well as the duration of certification. For example, in our model longer duration can actually result in more certification since it could provide stronger incentives to maintain quality (and only high-quality firms re-certify). Our model of certification as a costly information disclosure with timing chosen by the firm is similar to that in Schaar and Zhang (2015). In that

[4]Deviating firms could be either denied by the third-party certifier worried about creating a precedent in the industry and reducing the value of the certification program, or punished by expectations that once they certify sooner than expected, the market would expect them to certify even more often in the future. Such concerns for reputation for reticence or not revealing information too often are well known to managers in areas beyond certification. See for example Houston Lev and Tucker (2010) for voluntary earnings guidance by firms.

[5]See Mailath and Samuelson (2015) for a recent survey on the reputation literature.

[6]In some reputation models all equilibria are Markov, as shown in Feingold and Sannikov (2011) or Bohren (2016), but as we show here, focusing on MPEs sometimes leads to paradoxical results

paper quality is fixed so the firm certifies at most once and the focus of that paper is not on incentives to invest in quality but on the interplay between exogenous public news and endogenous certification.

Our paper is also somewhat related to the recent literature on reputation with information acquisition. (see e.g., Liu (2011)), where it is the buyers who can acquire information about the firm. The main difference is that in our model quality is endogenous and persistent, and it is the firm that incurs costs to provide information. Our model shares some features with the statistical discrimination literature initiated by Arrow (1973).[7] The underinvestment problem described in this paper is driven by the unobservability of quality and investment choices. The return to investment depends on the profits that the firm can assure by certifying high quality. In turn, these profits are determined by the buyers' expectation about past investments. In some sense, investment, certification, and buyers' beliefs are strategic complements, so that underinvestment becomes a self-fulfilling prophecy and an industry standard can help the firms and customers coordinate on equilibria with stronger incentives to invest.

The remainder of the paper is organized as follows. In Section 2 we describe the model. In Section 3, we study equilibria when the firm chooses when to certify based on its current reputation. We contrast this case with the optimal perfect Bayesian equilibria in Section 4 and discuss the implications for the optimal patterns of certification, investment and reputation.

2 Model

There is one firm and a competitive market of identical consumers. Time $t \in [0, \infty)$ is continuous. At every time t, the firm chooses privately investment in quality, makes decision about certification, and sells a product to the consumers, whose demand depends on perceived quality (firm's reputation).

We borrow the model of investment in quality from Board and Meyer-ter Vehn (2013). In particular, at time t, the firm's product quality is denoted by $\theta_t \in \{L, H\}$ where we normalize $L = 0$ and $H = 1$. Initial quality is commonly known to be low, $\theta_0 = L$, but subsequent quality depends on investment and unobservable technology shocks. Shocks are generated according to a Poisson process with arrival rate $\lambda > 0$. Quality θ_t is constant between shocks and is determined by the firm's investment at the most recent technology shock $s \leq t$ that is, $\theta_t = \theta_s$ and $\Pr(\theta_s = H) = a_s$. The firm observes product quality and

[7] See Arrow (1998) for a review of this literature.

chooses an investment plan $a = \{a_t\}_{t\geq 0}$, $a_t \in [0,1]$ which is predictable with respect to the filtration generated by $\theta = \{\theta_t\}_{t\geq 0}$. Investment has a marginal flow cost $k > 0$. Consumers observe neither quality nor investment. We denote their belief about the firm's investment by $\tilde{a} = \{\tilde{a}_t\}_{t\geq 0}$.

This specification implies that, given an investment policy a, quality jumps from L to H at an exponential time with arrival rate λa_t and jumps from H to L at a rate $\lambda(1 - a_t)$. As a consequence, investment has a persistent effect on product quality, as in the case when investment refers to employee training.[8]

Since λ measures the likelihood of shocks, a higher λ can be interpreted as capturing the instability of the firm's economic environment. On the technical side, note that since we assume $a_t \in [0,1]$, in the absence of investment the product quality can only experience negative shocks, and when investment is maximal, product quality can experience only positive shocks.

To focus on the role of certification in reputation, unlike Board and Meyer-ter Vehn (2013), we assume that there are no public signals about firm quality. Instead, the firm has access to an external (unmodeled) party, referred to as the certifier, who can credibly certify the current quality of the firm for a fee c. Product quality becomes public information at the time of certification.

We denote the firm's certification strategy by $d_t \in \{0,1\}$ and the market beliefs by $\tilde{d}$. The firm is risk neutral and discounts future payoffs at rate $r > 0$. We model the market in a reduced form by assuming that the firm profit flow is a linear function of its reputation, p_t, where $p_t = E^{\tilde{a},\tilde{d}}[\theta_t|\mathcal{F}^d_t]$ and $\mathcal{F}^d_t$ is the information generated by the firm's observed certification choices.

There are multiple ways to interpret this specification of profits. For example, as in Board and Meyer-ter Vehn (2013) the firm may be selling a limited amount of the product per period and the customers compete for the supply in a Bertrand fashion, which leads to prices being equal to the expected value of the product flow. Alternatively, the price may be fixed and the demand for the product may be proportional to the firm's reputation.

Given the firm's investment and certification strategy (a, d) and the market's belief about

[8]Also a retention and selection policy for employees has persistent effects on the quality of the workforce of a firm.

them $(\tilde{a}, \tilde{d})$, the firm's expected present value equals

$$E^{a,d,\theta_0}\left[\int_0^\infty e^{-rt}\big(p_t - a_t k\big)dt - \sum_{t\geq 0} e^{-rt}\, c\cdot d_t\right]$$

The conjectured investment and certification process $(\tilde{a}, \tilde{d})$ determines the firm's profit flow for a given history, while the actual strategy (a, d) determines the distribution over quality and histories.

Definition 1. *An equilibrium is a pair of strategies (a, d), and beliefs $(\tilde{a}, \tilde{d})$ such that given the market beliefs, the firm's strategy is optimal and beliefs are correct on the equilibrium path.*

Characterizing equilibria, throughout the paper we focus on pure strategy equilibria, in particular, in which the firm certification strategy, d, is pure.

Before studying the equilibrium, note that in the absence of disclosure, the evolution of reputation is given by the ordinary differential equation

$$\dot{p}_t = \lambda\big(\tilde{a}_t - p_t\big). \tag{1}$$

When $\tilde{a}_t = 0$ the reputation p_t drifts downward, and when $\tilde{a}_t = 1$, it drifts upward. Throughout the paper, we assume that k is sufficiently small, $k < \frac{\lambda}{\lambda+r}$. This implies that $a_t = 1$ is the first best investment, namely the investment the firm would choose if either quality or investment were observable by the market.

There are several possible histories off-the-equilibrium path: the firm may certify sooner than expected, in which case we assume consumers believe the certification is truthful (so that beliefs have to re-set to $p_t = 1$). Moreover, the firm may fail to certify even if it is believed to have maintained high quality by investing $a_t = 1$. In that case, the beliefs are not restricted by the Bayes' rule.

In what follows, we study equilibria in two classes. First, in Section 3, we consider belief-contingent (Markov perfect) equilibria in which the investment and certification strategies depend on reputation and quality. Later, in Section 4, we consider equilibria in which the strategies depend on the complete history.

Remark. We assume that voluntary certification is the only way customers reward firms for providing high quality. In some industries, there are other more important mechanisms. For example, warranties are a common way to reduce the moral hazard problem, as is the threat of losing repeated customers of experience goods. Moreover, as we described in footnote 1, there are other sources of information that affect firm reputation. In our opinion in several important industries voluntary certification plays a first-order role (as the examples in the beginning of the introduction suggest). One of the reasons is that verifying in court customer satisfaction may be expensive or impossible in such markets, so that warranties are impractical (as they appear to be in the markets for HMOs, child care and many dimensions of supplier relationships). Another reason is that many customers have one-off or rare transactions with the firm in such markets, so that dynamic threats of losing business if quality turns out to be low offer low-powered incentives. The co-existence of information coming from certification and third parties (e.g., word-of-mouth or reviews) seems to be more relevant to these markets. While we think that many of the economic effects identified in this paper are important also in a model with both certification and third-party information, a proper analysis of such a model is beyond the scope of this paper.

3 Markov Perfect Equilibria

In this section, we consider (pure strategy) Markov perfect equilibria. That is, we study equilibria in which the firm strategy (a, d) is a function only of its current quality θ and reputation p, and not of the full history of the game (in particular, it does not depend on the firm's actions before the last certification since we assume that every certification resets beliefs to $p_t = 1$ and recall that throughout the paper we restrict attention to pure certification strategies). Market expectations about firm certification and what investment strategies are hence a function only of reputation p.

Whenever the firm is expected to certify ($\tilde{d}(p) = 1$) the continuation value, $V_\theta(p)$, satisfies

$$V_H(p) = \max\big\{V_H(1) - c, V_H(0)\big\}. \tag{2}$$

on the other hand, when the firm is not expected to certify ($\tilde{d}(p) = 0$), the continuation value that satisfies the HJB equation

$$0 = \max_{a\in[0,1]} \; p - ak + \lambda(\tilde{a}(p) - p)V_L'(p) + \lambda a D(p) - rV_L(p) \tag{3}$$

$$0 = \max\Big\{ \max_{a\in[0,1]} \; p - ak + \lambda(\tilde{a}(p) - p)V_H'(p) - \lambda(1-a)D(p) - rV_H(p), \tag{4}$$
$$V_H(1) - c - V_H(p)\Big\},$$

where we call $D(p) \equiv V_H(p) - V_L(p)$ the value of quality, namely the gain the firm experiences when its quality improves, given reputation p.

The first step is to analyze the certification strategy. Whenever the market expects the high quality firm to certify we have that reputation jumps down to zero in absence of certification. Hence, the firm has two options: (i) certify and get a continuation value $V_H(1) - c$, (ii) do not certify and get a continuation value $V_H(0)$. Equation (2) says that the continuation value is the maximum between these two alternatives.

On the other hand, whenever the firm is not expected to certify, beliefs evolve according to Equation (1). If the firm certifies, its net gain (loss) is $V_H(1) - c - V_H(p)$; hence, the firm has incentives to certify if and only if

$$V_H(p) \leq V_H(1) - c.$$

In other words, the firm certifies whenever the gain caused by certification outweighs the (lumpy) certification cost. Whenever $V_H(p) > V_H(1) - c$, so the firm does not certify, the continuation value satisfies the differential equation

$$rV_H(p) = \max_{a\in[0,1]} \; p - ak + \lambda(\tilde{a}(p) - p)V_H'(p) - \lambda(1-a)D(p) \tag{5}$$

The economic intuition behind Equation (5) is the following: the flow continuation value, $rV_H(p)$, has three parts: i) the current profit flow, ii) the capital gains from changes in market beliefs (that affect future profit flows), and iii) the potential capital gains or losses from changes in privately known quality.

The next step is to analyze the firm investment decision, which is determined by the value of quality. From the HJB equation, we find that the firm's optimal investment policy is:

$$a(p) = \begin{cases} 0 & \text{if } \lambda D(p) < k \\ 1 & \text{if } \lambda D(p) > k, \end{cases}$$

and any a is optimal when $\lambda D(p) = k$, because the net present value of the investment is zero at that point. Note that due to our technological assumptions, the firm's investment incentives are independent of the state θ: investment increases the probability of a positive shock when the state is low and reduces the probability of a negative shock when the state is low, but in both cases the marginal benefit of investment is the same. This symmetry of investment allows us to write the equilibrium investment strategy as a function of market beliefs alone, $a(p)$.

Trivially, if the firm cannot communicate quality to the market, the value of quality is zero, $D(p) = 0$, leading to zero investment, $a = 0$. By contrast, if the information about quality were public, the firm would fully internalize the benefit of investment, leading to first best levels (i.e., $a = 1$). So unlike standard disclosure models (such as Dye (1985); Jovanovic (1982)) information has social value; it allows the firm to sustain investment and maintain a high level of quality. This is thus precisely the setting where certification could play a positive role by improving investment efficiency. Indeed, in static settings, Albano and Lizzeri (2001) demonstrate that certification plays a positive role, even when the certifier has monopoly power. We next show that this result does not hold in our (dynamic) setting even when the certification cost is arbitrarily small, at least as long as certification is based on current reputation.

To understand the link between the certification strategy and the investment incentives, observe that the evolution of the value of quality when the firm is not certifying is given by

$$rD(p) = \lambda(\tilde{a}(p) - p)D'(p) - \lambda D(p). \tag{6}$$

Let $p_c = \sup\{p \geq 0 : d(p, H) = 1\}$ be the highest reputation at which the high type decides to certify and let $\tau_c = \inf\{t > 0 : p_t = p_c, p_0 = 1\}$ be the time that it takes to reach this reputation. Since $\dot{p}_t = \lambda\big(\tilde{a}_t - p_t\big)$, we can integrate (6) over time to get that for any $t \in [0, \tau_c]$, or equivalently for any $p \in [p_c, 1]$, the value of quality at time t is:

$$D(p_t) = e^{-(r+\lambda)(\tau_c - t)} D(p_c). \tag{7}$$

So the value of quality deteriorates following the last certification. Certification has long lasting effects on reputation because quality is persistent. In turn, the firm has the weakest incentive to invest right after it certifies high quality (something an observer may call "resting on its laurels").

Furthermore, at the time/reputation the firm certifies, the value of quality is:

$$D(p_c) = V_H(p_c) - V_L(p_c) = V_H(1) - c - V_L(p_c).$$

Naturally, if the firm does not certify at time $t = \tau_c$, then the market infers that quality is low $\theta_{\tau_c} = L$, and, as a consequence, reputation drops to zero and remain at that level until the firm re-certifies. Therefore, $V_L(p_c) = V_L(0)$.

Our first lemma, shows that any equilibrium with positive benefits of certification can be characterized by two thresholds p_a and p_c such that the firm never invests before the certification time.[9]

Lemma 1. *Any pure strategy Markov perfect equilibrium is equivalent to an equilibrium defined by two thresholds p_a and p_c such that: $p_a \leq p_c$, $a(p) = 0$ if $p > p_a$ and $d(p, \theta) = \mathbf{1}_{\{p \leq p_c, \theta = H\}}$.*

This is a stark result. First, it implies that in any equilibrium where certification strategy is contingent on reputation, the firm either never invests in quality or only invests when reputation is at the lowest level. Second, it implies that the firm never invests in quality while its reputation is above the certification threshold. This combined with the Bayesian updating by the market implies that the firm invests, if at all, only when the market knows with certainty its product to be of low quality.

We provide a detailed proof in the Appendix, but here is the economic intuition. Suppose the firm has just certified so $p = 1$. If the firm is expected to fully invest in quality at some belief p_a, before the belief drops to p_c (i.e. if $p_a > p_c$), then the market belief would never cross p_a (recall that $\dot{p}_t = \lambda\big(\tilde{a}_t - p_t\big)$). But if so, the market belief would never drop to the certification threshold and we get a contradiction: a firm that is never expected to certify has no incentives to invest at all. [10]

With this result at hand, we can further characterize the equilibria. Since $V_L(0)$ equals the discounted expected gain derived from a positive quality shock, net of both the investment

[9] Formally, we say that two equilibria $(\hat{a}, \hat{d})$ and (a, d) are equivalent if $(\hat{a}_t, \hat{d}_t, \hat{\theta}_t) = (a_t, d_t, \theta_t)$ a.s., each t, where $\hat{\theta}$ and θ are the quality processes induced by the investment strategies $\hat{a}$ and a, respectively.

[10] As we show in the proof, even if the firm at p_a chooses an interior level of investment by (7) at slightly lower beliefs it would have strict incentives to put full investment, leading to the same contradiction

costs required to enable such a shock and the certification expense required to communicate to the market that quality increased, we have

$$V_L(0) = \frac{\lambda a(0)(V_H(1) - c) - a(0)k}{r + \lambda a(0)}. \tag{8}$$

If $p_c > 0$ (so that there is certification in equilibrium), then since failing to certify at p_c makes the market update that the quality is low, $V_H(p_c) = V_H(0) = V_H(1) - c$. Therefore, the value of quality at $p = p_c$ is

$$D(p_c) = D(0) = \frac{r\big(V_H(1) - c\big) + a(0)k}{r + \lambda a(0)}.$$

This expression allows us to fully characterize the set of equilibria. Lemma 1 implies that, in any equilibrium, the firm has at most weak incentives to invest. Hence, in any equilibrium with positive investment we have

$$D\left(p_c\right) = D(0) = \frac{k}{\lambda}.$$

Because the firm is indifferent about the level of investment, the continuation value at $p = 0$ can be computed assuming that $a = 0$. This yields the boundary condition

$$V_L(0) = V_L(p_c) = 0. \tag{9}$$

Similarly, we can also compute the continuation value assuming that $a(0) = 1$. If we combine Equations (8) and (9) we find that

$$V_H(p_c) = V_H(1) - c = \frac{k}{\lambda}. \tag{10}$$

Using these boundary conditions, we can solve for the continuation value in the no-disclosure region $(p_c, 1]$ and determine the disclosure threshold p_c. The next proposition characterizes the equilibrium.

Proposition 1. *In any Markov perfect equilibrium,*

(i) There is investment only if $p_t = 0$.

(ii) The payoff of a low quality firm is zero when $p_t = 0$. *That is,* $V_L(0) = 0$.

(iii) The payoff a high quality firm when $p_t = 1$ *is lower than the payoff if certification is unavailable. That is,* $V_H(1) \le 1/(r + \lambda)$.

In particular, the set of pure strategy Markov perfect equilibria is characterized as follows:

(i) If $c < \frac{1}{r+\lambda} - \frac{k}{\lambda}$, *then, there is an interval* $\mathcal{P}_c = [p_c^-, p_c^+]$ *of equilibrium certification thresholds. The lower threshold is given by*

$$p_c^- \equiv \left[1 - \frac{c}{\frac{1}{r+\lambda} - \frac{k}{\lambda}}\right]^{\frac{\lambda}{r+\lambda}},$$

and the upper threshold is the unique equilibrium threshold in which the zero profit condition $V_H(1) = c$ *holds.*

In any equilibrium with $p_c > p_c^-$ *the firm never invest, that is* $a(p_t) = 0$. *On the other hand, when* $p_c = p_c^-$ *we have that for any* $a^* \in [0, 1]$, *there is an equilibrium in which the high quality firm certifies whenever* $p_t \le p_c^-$ *and invests* $a(p_t) = a^* \mathbf{1}_{\{p_t=0\}}$. *The firm's payoffs are the same in all the equilibria with positive investment and are given by*

$$V_L(p_c) = 0$$

and

$$V_H(1) = \frac{k}{\lambda} + c.$$

(ii) If $\frac{1}{r+\lambda} - \frac{k}{\lambda} \le c \le \frac{1}{r+\lambda}$, *then the firm never invests and there is an interval* $\mathcal{P}_c = [p_c^-, p_c^+]$ *such that for any* $p_c \in \mathcal{P}_c$ *there is an equilibrium such that a high quality firm certifies whenever* $p_t \le p_c$. *The equilibrium with* $p_c = p_c^+$ *is the unique equilibrium in which the zero profit condition* $V_H(1) = c$ *holds, while* $p_c = p_c^-$ *is the unique equilibrium in which the smooth pasting condition* $V_H'(p_c) = 0$ *holds.*

(iii) If $c > \frac{1}{r+\lambda}$ *there is a unique equilibrium in which the firm neither invests nor certifies.*

The equilibrium taxonomy depends on the cost of certification. Naturally, for very high values of c, the equilibrium entails no disclosure and hence zero investment. When costs are intermediate, there is certification, but no investment. The most interesting case, however, is when the costs are low, so in what follows we assume that c is low enough that positive investment can be supported. Specifically, we assume that $c < \frac{1}{r+\lambda} - \frac{k}{\lambda}$.

Perhaps the most surprising observation in Proposition 1 is that in our model certification is practically unable to mitigate the firm's under-investment problem. Even in the equilibria that have the most investment in quality, the return to investment is zero (when the firm makes the investment in quality it is indifferent between putting positive investment and zero investment). Moreover, investment in quality happens only when the firm is known to have the lowest possible quality.

The intuition for that result is as follows. As we argued (in Lemma 1) in equilibrium the firm could only invest in quality when its reputation is the lowest. But why is the return to investment at that point zero? The reason is that if the firm had strict incentives to invest in quality at $p = 0$, then it would also have strict incentives to invest before reaching p_c (since $D(p_c) = D(0)$ and $D(p)$ is continuous in p for $p > p_c$). But then we would get the same contradiction as in Lemma 1: reputation would never reach the certification threshold and the firm would actually have no incentives to invest. Second, this indifference implies $V_L(0) = 0$: since the firm has at most weak incentives to invest in quality at $p = 0$, its equilibrium payoff can be computed by using the strategy of never investing.[11]

While that result is very stark, it is not the main takeaway of Proposition 1. Instead, the main takeaway is the existence of MPE with very high frequency of certification, no investment and very low payoff to the high quality firms, as low as $V_H(1) = c$. This is the over-certification trap we discussed before. The existence of such equilibria appears very robust. It extends to a model with additional public news and a more general quality transition process. The intuition is that as long as the firm knows its quality, if the market expects it to re-certify frequently, the firm may find it very difficult to convince the buyers that it delays certification because it wants to get out of the trap and not because it has failed to maintain high quality. A high enough certification frequency can be chosen to dissipate most of the gains from reputation and thereby reduce or remove incentives for investment.

As we show in the next section, while the low-payoff-no-investment MPEs appear quite robust even for low costs of certification, there exist equilibria with investment and high payoffs. Therefore, an industry standard or other ways to coordinate on better equilibria can be very effective in improving firm's payoffs and overall efficiency.

Remark. The result that all MPEs have no investment until the reputation drops to zero depends on our assumption that quality can only improve if the firm chooses full investment.

[11]This helps explain two stark consequences of Proposition 1 for equilibria with positive investment. The ex-ante payoff of the high-quality firm is increasing in the certification costs and costs of investment, k. The high-quality firm is better off when the certification is more expensive and investment is more costly! The intuition is as follows. The frequency of certification must be high enough to dissipate enough profits so that $V_H(1)$ is low enough that the L type is indifferent between investing and not investing at $p = 0$. The higher c or k, the less attractive is investment to the low type, so the certification needs to be less frequent to keep it indifferent (notice that p_c decreases in k). That helps the high type.

For example, if instead quality jumped from H to L at a rate $\lambda(1 - a_t * (1 - \epsilon))$ for some small ϵ, then for small costs of certification there would exist MPEs with investment for all t. Roughly, in such an MPE, right after successful certification, reputation deteriorates slowly from $p_0 = 1$ despite the belief that the firm chooses $a_t = 1$. It is then possible to pick p_c in a way to economize on certification costs while still maintaining incentives for $a_t = 1$. Such equilibria are very similar to the time-based equilibria we discuss in Marinovic et al. (2016). Still, even in that model there is a large range of MPE including the ones with no investment and very low payoffs for the high quality firms, as the ones described in Proposition 1. Moreover, these MPEs can be further improved upon by PBE similar to the ones described in the next section, so the benefits of coordination are robust.

One can also use our characterization of equilibria to revisit the natural question of pricing of certification. Consider the equilibria with the most efficient investment. From the point of view of the firm, cheaper certification is offset by the equilibrium effect that the market expects it to certify more often. The latter effect dominates, making the firm worse off as c decreases. A profit-maximizing certifier faces a downward-sloping demand curve: lower c leads to more frequent certification. If the marginal cost of the certifier is close to zero (the cost of providing additional certification), we expect the optimal price to be very low. To see this, consider the extreme case of zero marginal cost. Then, as c goes down, certification and hence investment are more frequent. Since paying c is just a transfer, the overall efficiency increases. At the same time, the profits of the firm go down, which implies that the profit of the certifier goes up as well. Hence the certifier profits go up as c decreases towards zero (the limit revenues are positive since the frequency of certification goes to infinity). This tendency to set low fees to benefit from more frequent certification adds a new consideration to our standard intuition from the static model in Lizzeri (1999).

In our dynamic context the certification inefficiency is exacerbated when the cost of certification vanishes. Indeed, the present value of expected certification expenses increases as the certification cost vanishes, because the frequency of certification increases as well. A priori, one could hope that the best MPE converges to first best when c goes to zero. As we have shown this is not the case and one of the reasons is that the frequency of certification increases faster than the reduction in the cost; hence, the present value of future certification costs does not go to zero. However, this is not the only reason of why the limit is not efficient. Even if the cost where just a transfer that doesn't affect overall welfare, the equilibrium would not converge to first best. The reason is that even in the limit investment is still inefficient. In the first best there is constant full investment; however, in any MPE with investment, a high quality firm never invest and a low quality firm only invest when it is known to be low quality. In the limit when c goes to zero, type is known by the market effectively at every time, but investment remains inefficient. We summarize this discussion in the following corollary:

Corollary 1. *In the limit when $c \to 0$ the equilibrium outcome converges to $p_t = \theta_t$ and $a_t > 0$ if and only if $\theta_t = L$.*

Proof. The result follows from the characterization of the equilibrium in Proposition 1 and the observation that the disclosure threshold p_c converges to 1 when c goes to zero so the set of disclosure times in the limit is dense in $\mathbb{R}^+$. □

4 Optimal Perfect Bayesian Equilibria

As we mentioned in the Introduction, it is common in the literature on dynamic reputation to interpret voluntary information disclosure without commitment as corresponding to Markov perfect equilibria of the game that we studied in the previous section. Our interpretation of the results in the previous section is that they suggest that without an industry standard (or some other third-party coordination) it may be hard for individual firms to reap much benefits from voluntary certification, or that they can dissipate most or even all value of reputation by excessive certification. In fact, the previous section showed that voluntary certification without (implicit or explicit) commitment to coordinate consumer expectations and firm actions, results in too much certification, too little investment in equilibrium and no net benefits for low-quality firms entering the market.

To model an industry standard that coordinates firms and customer expectations, we now look at non-Markov equilibria. In this section, we study the best Perfect Bayesian Equilibria of our game. We show that even if the industry standard cannot impose fines nor bonuses and can only announce a time schedule for expected certifications and re-certifications of high-quality firms, it can result in vastly better outcomes for the firms. We also provide insights about the features of optimal industry standards, showing that not only higher payoffs can be achieved, but also that the optimal standard (the strategy in the optimal equilibrium) has quite natural and realistic features.

We exploit the recursive nature of the problem to analyze the set of equilibrium payoffs. Since in our game the firm has private information about its type that changes over time, it is not a repeated game. Yet, because certification perfectly reveals high type, there is no external news in between certifications, and we look at equilibria in pure certification strategies, we can use the times at which the high quality firm certifies on the equilibrium path to define a regenerative process. We can then use this regenerative process to factorize the equilibrium payoffs using a procedure analogous to that in Abreu, Pearce and Stacchetti (1990) (hereafter, APS).

We start by introducing some notation. Let $d_t(H) \in \{0,1\}$ be the equilibrium certification decision at time t conditional on $\theta_t = H$. Define the sequence of times $T_n = \inf\{t > T_{n-1} : d_t(H) = 1\}$, $T_0 = 0$ recursively (T_{n+1} can depend on the public history up to T_n). In equilibrium, a high quality firm certifies at time T_n so $p_{T_n} = 1$ if $\theta_{T_n} = H$. A low quality firm does not certify at this time and this is interpreted as perfect evidence that the firm has low quality, i.e., $p_{T_n} = 0$ if $\theta_{T_n} = L$. Accordingly, on the equilibrium path there is a common belief about the firm quality at each T_n. This means that the set of continuation payoffs at time T_n, $n \geq 0$, only depends on θ_{T_n} and not the whole history of the game. Hence, with the addition of a public randomization device, the set of continuation equilibria is the same at every T_n.[12] Therefore, in order to characterize the equilibrium payoff set we can use the tools from APS and decompose any equilibrium into current strategies and continuation values after public signals generated by certification (which in our setting is the only source of public signals).

To proceed with this recursive characterization, it is convenient to measure the time elapsed since T_{n-1}. Hence, for any date $s \in [T_{n-1}, T_n]$, we let $t = s - T_{n-1}$ and $\tau = T_n - T_{n-1}$. The continuation value at time t is denoted by $U_{\theta_t}(t|\theta_0)$ (it depends on the quality at the last certification date, θ_0, and the current θ_t known by the firm). Adapting the APS approach, we factorize the firm's payoff using the time τ when a high quality firm certifies for the first time, the investment strategy up to time τ, and the continuation value given the certification decision at time τ.

Let's denote the worst and the best equilibrium payoffs of a type θ_0 at $t = 0$ (that is, at the date T_{n-1}) by $\underline{U}_{\theta_0}$ and $\overline{U}_{\theta_0}$, respectively. The worst payoffs have to be individually rational for the firm, and we can use the Markov equilibria in Proposition 1 to determine the worst payoff for either type. In particular, the worst Markov perfect equilibria minimax the firm payoffs, so that $\underline{U}_H = c$ and $\underline{U}_L = 0$.[13]

[12]The randomization device is needed for this claim since otherwise past outcomes could be used to coordinate on continuation play. As we show later, the optimal equilibria we construct do not use the randomization device.

[13]At $t = 0$ the high-quality firm just incurred cost c to certify. Hence, its continuation payoff has to be at least c since otherwise it would deviate at T_{n-1}.

By the standard bang-bang property, without loss of generality, we can focus attention on equilibria with continuation payoffs that randomize at τ over $\{\underline{U}_{\theta_0}, \overline{U}_{\theta_0}\}$ based on the firm certification choice at that time. In principle, there are two such randomizations to consider: when the firm certifies and when it does not. When the firm certifies, continuing with the best equilibrium is good for on-path expected payoffs and for incentives to invest. So the equilibrium with the highest ex-ante payoff continues to $\overline{U}_H$ when the firm certifies. Therefore, to describe continuation strategies for the best equilibrium if we start with type θ, we only need to specify probability β of transitioning to $\underline{U}_L$ (a punishment phase corresponding to the worst equilibrium) if the firm fails to certify at τ.

The firm's incentives to invest at t are determined by the value of quality given, as before, by the difference $D(t|\theta_0) \equiv U_H(t|\theta_0) - U_L(t|\theta_0)$. For any $t \in [0, \tau)$, the continuation values satisfy HJB equations analogous to the Markovian case:

$$0 = \max_{a\in[0,1]} \; p_t^{\theta_0} - ak + \dot{U}_L(t|\theta_0) + \lambda a D(t|\theta_0) - rU_L(t|\theta_0) \tag{11}$$

$$0 = \max_{a\in[0,1]} \; p_t^{\theta_0} - ak + \dot{U}_H(t|\theta_0) - \lambda(1-a) D(t|\theta_0) - rU_H(t|\theta_0), \tag{12}$$

where $p_t^{\theta_0}$ is the value of p_t given $p_0 = \theta_0$. As we did in the analysis of the Markov perfect equilibrium, we can integrate these HJB equations between time t and τ to get

$$D(t|\theta_0) = e^{-(r+\lambda)(\tau-t)} D(\tau|\theta_0). \tag{13}$$

A direct consequence of equation (13) is that incentives to invest are increasing in time. The firm optimal investment policy is to invest as soon as $D(t|\theta_0) \geq k/\lambda$, this means that investment strategy is fully characterized by the time τ_a at which this incentive compatibility constraint is satisfied, and can be written as $a_t = \mathbf{1}_{t>\tau_a}$. That the investment strategy is completely determined by $D(\tau|\theta_0)$ turns out to be quite useful. Given $(\tau_{\theta_0}, \beta_{\theta_0}, \underline{U}_{\theta_0}, \overline{U}_{\theta_0})$, the firm's optimal investment strategy (described by τ_a) depends deterministically on $D(\tau|\theta_0)$ which equals:

$$D(\tau_{\theta_0}|\theta_0) = \overline{U}_H - c - \left(\beta_{\theta_0}\underline{U}_L + (1-\beta_{\theta_0})\overline{U}_L\right) = \overline{U}_H - c - (1-\beta_{\theta_0})\overline{U}_L.$$

The previous equation shows that, for a given set of continuation payoffs and for a given starting type θ_0, once we specify τ and β, the firm's investment policy is uniquely determined by the incentive compatibility constraints and so is the total payoff from this equilibrium.

In other words, given $(\underline{U}_{\theta_0}, \overline{U}_{\theta_0})$, the best equilibrium is fully characterized by two pairs $(\tau_L^*, \beta_L^*), (\tau_H^*, \beta_H^*)$ that are the times to next certification opportunity and the punishment probability at that time that depend on the market belief about firm quality at the last time of possible certification (or the beginning of the game). Therefore, to find the optimal equilibrium, we only need to optimize over $(\tau_\theta, \beta_\theta)$ and we can do this by computing the firm's payoff as:

$$\mathcal{U}_{\theta_0}(\tau, \beta) \equiv \int_0^\tau e^{-rt}(p_t^{\theta_0} - \mathbf{1}_{t \geq \tau_a} k)dt + e^{-r\tau}\left(p_\tau^{\theta_0}(\overline{U}_H - c) + (1 - p_\tau^{\theta_0})(1-\beta)\overline{U}_L\right).$$

Thus, we have reduced the problem of finding the best equilibrium to solving the following optimization problem (for a given set of continuation payoffs)

$$\overline{U}_{\theta_0} = \max_{\tau \geq 0, \beta \in [0,1]} \mathcal{U}_{\theta_0}(\tau, \beta). \tag{14}$$

Now, strictly speaking, this is a relaxed problem because there are two incentive compatibility constraints that we have ignored so far: (1) a high quality firm does not certify before time τ, and (2) a high quality firm does not "skip" the opportunity to certify at time τ. We can ignore (1) because we can always attach continuation payoff $\underline{U}_H = c$ if the firm certifies when it is not supposed to do so (so, before it spends c for certification it gets payoff 0). We ignore (2) for the moment and verify later on (in the proof of Proposition 2) that it is not optimal for a high quality firm to delay certification at time τ.

The next step in our analysis is to show that the optimal β_θ^* is either zero or one, so that the optimal equilibrium/best industry standard does not randomize when the firm fails to certify.

Lemma 2. *In the best equilibrium the probability β of triggering a punishment when the firm fails to certify at τ is either zero or one.*

When $\beta_L^* = 0$ we call the equilibrium *lenient* since failing to certify does not trigger punishment and the firm is given multiple opportunities to certify till it finally gets a success. When $\beta_L^* = 1$ we call the equilibrium *harsh* since after failing to certify the first time, the low-quality firm never certifies again. The proof of the lemma works as follows. We fix θ_0 and the investment level that we want to implement, τ_a, and look at the trade-off between β and τ. One way to analyze this trade-off is to look at the firm's payoff as we move along the "iso-incentive" curve in (in the plane (β, θ)) that implements that τ_a. By doing that, we show in the proof that the payoff is a convex function of β along this "iso-incentive" curve. This means that the solution for β is either zero or one.

Equation (14) points out that to find $\{\overline{U}_L, \overline{U}_H\}$, we need to solve a fixed point problem finding the two values at the same time. Luckily, we start with characterizing $\overline{U}_H$ and show that for small c it is independent of $\overline{U}_L$. It allows us to find $\overline{U}_H$ first and then use that value to solve for $\overline{U}_L$. The first step in the construction of the equilibrium is to characterize equilibria with full investment, and later show that for small c the best equilibrium has indeed full investment. With full investment, if $p_0 = 1$ and $\theta_0 = H$ then on path $p_t = 1$ and $\theta_t = H$, for all $t \in [0, \tau]$. This happens because with full investment quality never drops, so the payoff of a high quality firm simplifies to

$$U_H^{FI}(\tau) = \frac{1-k}{r} - \frac{e^{-r\tau}}{1-e^{-r\tau}}c. \tag{15}$$

Moreover, with full investment, once high quality is reached, any punishment for failing to certify is off-equilibrium path, and so it is optimal to use the harshest possible punishment, which corresponds to $\beta_H = 1$. In addition, among all the equilibria that implement full investment, the best one has the minimum amount of certification. The minimum frequency of certification that implements full investment requires that the incentive compatibility constraint binds at $t = 0$ (recall that incentives increase as we get closer to certification). Otherwise, we could reduce the cost of certification while still providing enough incentives. Hence, the best equilibrium implementing full investment given $\theta_0 = H$ and $\tau_a = 0$, which we denote by τ_H^{FI}, is implicitly defined by

$$e^{-(r+\lambda)\tau_H^{FI}}\left(U_H^{FI}(\tau_H^{FI}) - c\right) = \frac{k}{\lambda}. \tag{16}$$

Note that $U_H^{FI}(\tau_H^{FI})$ is independent of $\overline{U}_L$. So if indeed the best equilibrium $\overline{U}_H$ induces full investment, we can solve for the best equilibria in two steps. First, we solve for the best equilibrium when $\theta_0 = H$ and then we use this solution to solve for the best equilibrium at the outset of the game when $\theta_0 = L$. As part of the construction of the best equilibrium, we show that for small certification cost, the certification frequency given $\theta_0 = H$ is $\tau_H^* = \tau_H^{FI}$ and the maximum payoff is $\overline{U}_H = U_H^{FI}(\tau_H^*)$.

The next step is to characterize the best equilibrium payoff if we start with a low quality firm, $\overline{U}_L$, keeping fixed τ_H^* and $\overline{U}_H$. Without loss of generality, we can restrict attention to equilibria with full investment between time zero and τ.[14] The optimal certification frequency in the low state maximizes

$$\tau_L^* \in \arg \max_{\tau_L, \beta_L \in [0,1]} \int_0^{\tau_L} e^{-rt}(p_t^L - k)dt + e^{-r\tau_L}\left(p_{\tau_L}^L(\overline{U}_H - c) + (1 - p_{\tau_L}^L)(1 - \beta_L)\overline{U}_L\right) \tag{17}$$

subject to

$$e^{-(r+\lambda)\tau_L}\left(\overline{U}_H - c - (1 - \beta_L)\overline{U}_L\right) \geq \frac{k}{\lambda}.$$

At this point in the analysis, our bang-bang Lemma 2 provides a great simplification: in order to find the best equilibrium when $\theta_0 = L$, we only need to compare the payoff when $\beta_L = 0$ to the payoff when $\beta_L = 1$. For $\beta_L = 1$, the payoff of the firm can be computed directly and is given by

$$\begin{aligned} \hat{U}_L^1 &= \int_0^{\tau_L^1} e^{-rt}(p_t^L - k)dt + e^{-r\tau_L^1} p_{\tau_L^1}^L(\overline{U}_H - c) \\ \tau_L^1 &= \frac{1}{r+\lambda} \log\left(\frac{\lambda(\overline{U}_H - c)}{k}\right). \end{aligned} \tag{18}$$

For $\beta_L = 0$ some extra work is needed because the expected payoff is implicitly determined by the solution to the fixed point problem

$$\begin{aligned} \hat{U}_L^0 &= \int_0^{\tau_L^0} e^{-rt}(p_t^L - k)dt + e^{-r\tau_L^0}\left(p_{\tau_L^0}^L(\overline{U}_H - c) + (1 - p_{\tau_L^0}^L)\hat{U}_L^0\right) \\ \tau_L^0 &= \frac{1}{r+\lambda} \log\left(\frac{\lambda(\overline{U}_H - c - \hat{U}_L^0)}{k}\right). \end{aligned} \tag{19}$$

The certification time must be strictly positive, $\tau_L^0 > 0$, which means that the payoff $\hat{U}_L^0$ must be strictly lower than $\overline{U}_H - c - k/\lambda$. Once we have computed these two payoffs, the best equilibrium is given just by larger one and the probability of triggering a punishment is

$$\beta_L^* = \arg \max_{\beta \in \{0,1\}} \left\{(1-\beta)\hat{U}_L^0 + \beta \hat{U}_L^1\right\}.$$

[14]Suppose this is not the case and $\tau_a > 0$. If there is no investment between time zero and time τ_a then $\theta_{\tau_a} = L$ and $p_{\tau_a} = 0$. This means that the continuation game at time τ_a looks the same as at time zero. But then $\overline{U}_L = e^{-r\tau_a}U_L(\tau_a) < U_L(\tau_a)$ which cannot be the case as we can consider an alternative equilibrium in which the continuation equilibrium at time zero (calendar time T_n) is the same as the continuation equilibrium at time τ_a (calendar time $T_n + \tau_a$). The only other possibility is that there is no investment by the low quality firm in the best equilibrium, so that $\overline{U}_L = 0$, which we show by construction not to be true when c is small.

The next proposition, which characterizes the best equilibrium when the cost of certification is low, provides the main result of this section:

Proposition 2. *There exists $c^{max} > 0$ such that for any $c \leq c^{max}$, the best equilibrium payoffs $\overline{U}_H, \overline{U}_L$ are achieved in an equilibrium with the following features:*

(i) A regular phase in which:

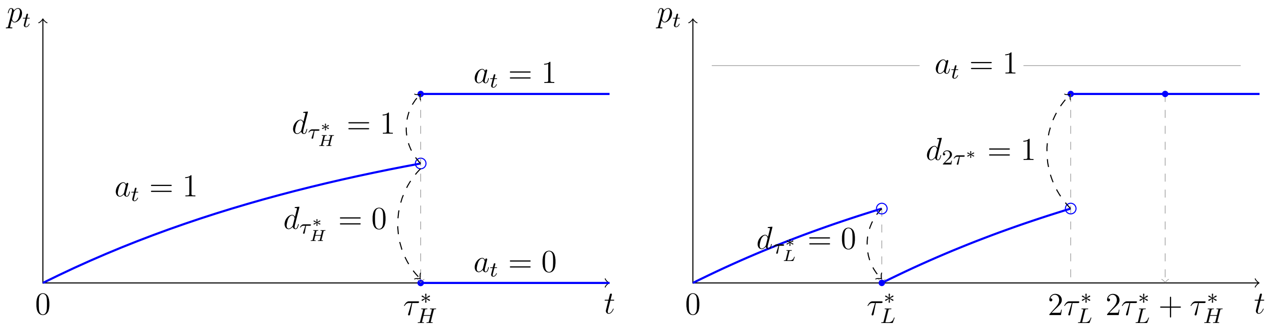
- *(a) There is full investment.*
- *(b) A firm that has certified in the past, is expected to certify at constant intervals of length $\tau_H^* = \tau_H^{FI}$. If such firm ever fails to certify a punishment phase starts (i.e. $\beta_H^* = 1$).*
- *(c) A firm that has never certified is allowed to certify every τ_L^* until the first success (i.e. $\beta_L^* = 0$).*

(ii) A punishment phase corresponding to the worst Markov perfect equilibrium in which there is no investment.

The equilibrium for small c is quite different for firms that have certified in the past than for new firms that have not been certified yet. Recall that by Lemma 2, the equilibrium can be either lenient or harsh. Proposition 2 shows that when the cost of certification is low, the equilibrium is lenient ($\beta_L = 0$) in the sense that new firms (that have never certified before) that fail to certify at the end of the probationary period (of length τ_L^*) are not excluded from future certification. Instead they are given a clean slate and another chance until they finally manage to reach the high quality. This is quite different for established firms that fail to certify: those are always and forever punished for failing to certify.

This result implies the following feature of the design of industry standards: industry certification should treat new firms and established firms (that have already certified in the past) quite differently. In particular, an industry certification agency should be harsher with established firms that have reduced their quality (which is detected when they fail to certify at τ_H^*) than with new entering firms. Of course this result hinges on the assumption that the main objective of the certification agency is to improve the overall quality in the industry (not taking into account any competitive effects assuming that it is possible to maintain high quality with a very high probability). If the main objective of the certification agency were to generate entry barriers then the industry standard would be probably harsher for new firms.

Proposition 2 provides the best equilibrium for small cost. In Figure 2 we present a numerical example that shows that if the cost of certification is high, the equilibrium may be harsh ($\beta_L = 1$). In this case, new firms are subject to a probationary period and if they fail to certify are in a sense excluded from the market. That is, after failing to certify for the first time we move to a Markov perfect equilibrium with no further investment. The harsh equilibrium is more likely for large c when the cost of investment k is small and λ is high (the additional condition on λ means that the probability of triggering the punishment on the equilibrium path is small). In the Appendix (section B.1), we show analytically that punishment, β_L, is non-decreasing in c. Figure 1 shows the dynamics of reputation, certification and investment in both kinds of equilibria. In the harsh equilibrium, the firm stops investing as soon as it fails to certify. On the other hand, in the lenient equilibrium, the firm never stops investing on the equilibrium path.



(a) Harsh Equilibrium **(b)** Lenient Equilibrium

Figure 1: Sample Path: Harsh vs Lenient Equilbrium

What determines the nature of the equilibrium? Figure 2 shows the comparative statics with respect to c. When the cost of certification is small, the best equilibrium is lenient, while the equilibrium is harsh when c is large (at least for some parameters). The nature of the equilibrium is determined by the following trade-off: a harsher punishment requires lower frequency of certification to provide incentives. This is particularly advantageous if c is large. The disadvantage is that there is a higher probability of triggering a punishment by mistake (even though the firm made the right investments, but was just unlucky in improving quality). The surplus destroyed by the punishment is decreasing in c, which means that the cost of triggering a punishment is lower when c is large. In sum, the net benefit of using harsher punishments is higher when c is large, which implies that it is optimal to punish new firms that fail to certify only if c is sufficiently large.[15]

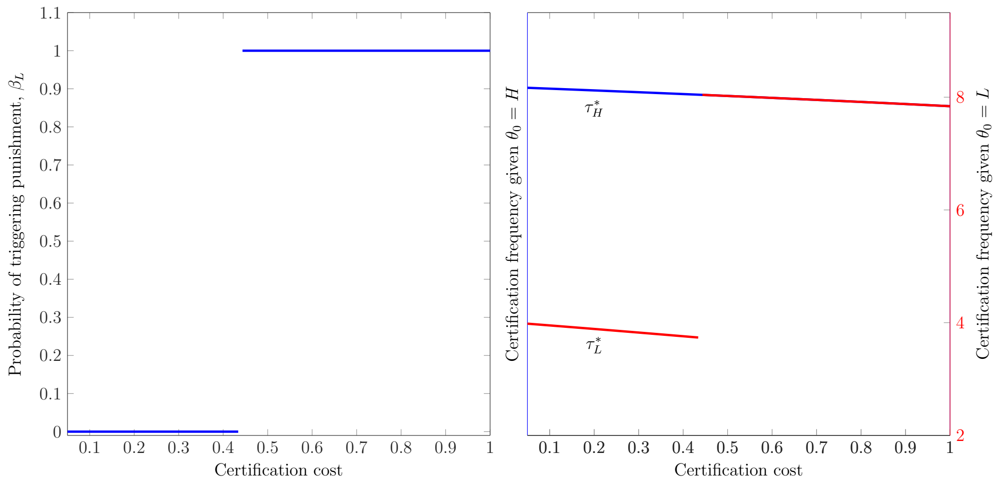


Figure 2: Effect of certification cost on best equilibrium. Parameters: $r = 0.05$, $\lambda = 0.5$, $k = 0.1$, $c \in [0.1, 1]$

Finally, note another feature of the optimal industry standard: when the firm starts with low quality, it is not allowed to certify as soon as the quality improves but has to wait till τ_L^* to do so. At first it may appear that we have not allowed for such a possibility, since we have constructed equilibria in which the certification time is deterministic (does not depend on the current θ_t). It may also appear wasteful that we force the firm to wait till τ_L^*. Yet, it turns out that it is indeed optimal to force the firm to wait. The intuition is that the firm revenue flow payoff p_t^L incorporates the possibility that the quality has changed before τ_L^*. If we allowed the firm to certify as soon as it gets high quality, p_t^L would be zero until such certification. Since market beliefs are correct on average, from the ex-ante point of view, the firm would not benefit in terms of revenues from early certification, but would only incur the certification costs sooner, which is suboptimal.[16] That said, since this cost is incurred only once in the whole game (as opposed to the costs after the firm reaches high quality), industry standards that would allow firms to certify for the first time as soon as they achieve high quality, would be approximately optimal.

[15] The previous discussion suggests that it could be the case that for large values of c, $\beta_H = 0$ is optimal. This can only be the case if the best equilibrium has less than full investment. Given the bang-bang nature of the equilibrium, we only need to compare the best equilibrium with $\beta_H = 0$ to the best equilibrium with $\beta_H = 1$. Extensive numerical computations suggest that the best equilibrium has either full investment (and so $\beta_H = 1$) or no investment at all (in which case $\tau_H = \infty$ and there is no certification).

[16] Technically, our analysis incorporates approximately that strategy since we allow τ_L^* arbitrarily close to zero and $\beta_L^* = 0$.

5 Final Comments and Discussion

In this paper we study voluntary certification as a mechanism used by firms to improve their reputation when quality and investment are unobservable. Our focus is on certification and investment incentives. We consider a dynamic setting in which a firm decides not only whether to certify, but also when. Unlike in most of the prior reputation literature, reputation depends on endogenous and voluntary disclosure instead of exogenous signals (for example, consumer reviews).

We show that whether voluntary certification manages to create the right incentives for investment, helps the firms reap benefits of such investment, and results in persistent rather than temporary reputations, depends on whether the industry manages to coordinate on a good certification standard. Since information about quality has to be provided by the firm itself, reputation depends on the market's expectations of when high quality firms should certify and the equilibrium can suffer from over-certification trap (which in turn creates under-investment). We contrast the efficiency of Markov perfect equilibria and optimal perfect Bayesian equilibria. One of the main lessons is that third party certification may have little ability to increase investment and actually become an unnecessary burden for the firms. Only well-designed systems that prevent the tendency to engage in excessive certification can lead to higher efficiency. Our analysis of the optimal perfect Bayesian equilibrium highlights some key aspects that an optimal certification (or licensing) standard must consider, such as the frequency of certification and the possibility of excluding firms that fail to certify.

The range of possible equilibrium outcomes seems to be consistent with market experience. For example, some certification systems have been criticized. In particular, despite its widespread use, the ISO process has been criticized as wasteful. Dalgleish (2005) cites the "inordinate and often unnecessary paperwork burden" of ISO, and asserts "managers feel that ISO's overhead and paperwork are excessive and extremely inefficient. Despite their dislike, many companies are registered. Firms maintain their ISO registration because almost all of their big customers require it." Our model sheds light on this apparent contradiction. Since the mere availability of certificates modifies market beliefs about uncertified firms, it can operate as a threat that destroys firm value by forcing firms to incur large costs to avoid the penalty (in terms of price or volume) the market applies to uncertified firms.

On the other hand, our analysis shows that certification can be an effective communication channel in industries that organize the certification process in a way that prevents the excessive use of certification. Firm dynamics are often driven by uncertainty regarding the quality of new products. For example, Atkeson, Hellwig and Ordoñez (2015) argue that "if it takes buyers time to learn about the quality of entering firms, these firms initially face lower demand and prices until they are able to establish a good reputation for their product." Even though licensing has been previously criticized as a way to increase barriers to entry, we show that if the main barrier to entry is consumers' uncertainty, then a well-designed industry certification standard can help reduce barriers by mitigating the effect of asymmetric information and moral hazard.

The best equilibria we characterized may in some situations call for commitment that an industry certifier may find hard to maintain: for example, low-reputation firms are not allowed to certify improvements in quality too early. If certification costs are small, equilibria that use much less commitment but yield very similar payoffs to the best equilibrium we characterized, can be constructed. For example, a reputation system in which high-quality firms have to re-certify at a constant time frequency and low-reputation firms can certify as soon as they improve quality achieves approximately the first-best payoffs if certification costs are low. See more details in Marinovic et al. (2016).

In this paper we have purposely ignored alternative sources of information that the market may use to learn about quality, notably public ratings (Ekmekci, 2011) and consumer reviews (Cabral and Hortacsu, 2010). By restricting attention to certification as the only information channel, we thus consider a clean setting for understanding the informational role of certification. In our setting information can have social value (since it can help improve investment in quality) and we seek to understand whether and when certification can deliver such value. In many markets certification is the main source of information about quality that the customers have and hence we think our model is applicable to such markets. In other markets customers learn both from reviews (or other outside news) and from voluntary certification. To understand such markets better, we think future research should analyze model combining these sources of information.

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Appendix

A Proofs Section 3

Proof Lemma 1

Proof. Let $p_a \equiv \sup\{p \in [0,1] : a(p) > 0\}$, $p_c \equiv \sup\{p \in [0,1] : d(p,H) = 1\}$, $\tau_a \equiv \inf\{t > 0 : p_t = p_a, p_0 = 1\}$, and $\tau_c \equiv \inf\{t > 0 : p_t = p_c, p_0 = 1\}$.

First, we show that in any equilibrium $p_a \leq p_c$. Looking for a contradiction, suppose that $p_a > p_c$. Let's consider the behavior of beliefs at the threshold p_a. If $a(p_a) \geq p_a$ then $\lambda(a(p_a) - p_a) \geq 0$ so beliefs never cross the threshold p_a. On the other hand, if $a(p_a) < p_a$ then beliefs cross the threshold p_a however if this is the case, we have that $k/\lambda = D(p_a) = e^{-(r+\lambda)(\tau_c-\tau_a)}D(p_c) < e^{-(r+\lambda)(\tau_c-t)}D(p_c) = D(p_t)$ for all $t \in (\tau_a, \tau_c]$. This means that $a(p_a - \epsilon) = 1$ but if this is the case then beliefs can never cross the threshold p_a. This in turn implies that $\tau_c = \infty$, so that $D(p_t) = e^{-(r+\lambda)(\tau_c-t)}D(p_c) = 0$. This contradicts the hypothesis that $p_a > p_c \geq 0$ which requires that $\lambda D(p_a) \geq k$.

Second, we analyze the certification strategy. By definition we have that $d(p,\theta) = 0$ for $p > p_c$ and $d(p_c, H) = 1$. If the firm fails to certify at time τ_c beliefs drop to zero so $p_{\tau_c^+} = 0$. The next step is to specify the certification strategy when $p_0 = 0$. We consider two cases: $V_H(1) - c > 0$ and $V_H(1) - c = 0$ ($V_H(1) - c < 0$ is trivial because in this case certification is suboptimal so $d_t = 0$). Let's consider the case with $V_H(1) - c > 0$ first. Suppose that $\tilde{p} = \inf\{p : d(p,H) = 1\} > 0$ and let $\tilde{\tau} = \inf\{t : p_t = \tilde{p}, p_0 = 0\}$. Using the incentives equation we have

$$D(0) = e^{-(r+\lambda)\tilde{\tau}}D(\tilde{p}).$$

By construction we have that $V_H(\tilde{p}) = V_H(1) - c = V_H(p_c) = V_H(0)$ (Note that it cannot be the case that $V_H(p_c) \neq V_H(0)$ as this would contradict the optimality of the certification strategy). Similarly, we also have $V_L(\tilde{p}) = V_L(0)$ because the market infers that the firm has low quality if it fails to certify when $p_t = \tilde{p}$. Thus, $D(\tilde{p}) = V_H(\tilde{p}) - V_L(\tilde{p}) = V_H(0) - V_L(0) = D(0) = D(p_c)$. Replacing this in the equation (7) we get

$$D(0) = e^{-(r+\lambda)\tilde{\tau}}D(0) \Rightarrow D(0) = 0.$$

If this is the case then we have that $a(p) = 0$ for all $p \in [0, \tilde{p}]$ and in particular $a(0) = 0$ so $(p_0 = 0, \theta_0 = L)$ is an absorbing state and $V_L(0) = 0$. This, together with $D(0) = 0$, implies that $V_H(0) = 0$, which contradicts the hypothesis $V_H(1) - c > 0$. Hence, it must be the case that $\tilde{p} \leq 0$.

Next, we consider the case with $V_H(1) - c = 0$. In this case, by a similar argument as the one used before, we have that $D(0) = 0$, so $a(0) = 0$ and $(p_t = 0, \theta_t = L)$ is an absorbing state. This means that for any strategy $\tilde{d}_t$ in which the low quality firm never certifies there is some threshold p_c such that $\Pr(\tilde{d}_t = \mathbf{1}_{\{p_t \leq p_c, \theta_t = H\}} | \theta_0) = 1$ for all $t \geq 0$. Moreover, the restriction to strategies in which the low type never certifies is without loss of generality as in equilibrium the low type would never find optimal to certify low quality. □

Proof Proposition 1

Proof. We need to analyze several cases depending on the cost of certification and whether we have investment in equilibrium or not. In absence of investment we have that quality starts at $\theta = H$, it depreciates at a rate λ and $\theta = L$ is an absorbing state. The first set of results characterizes the value function when this is the case.

Equilibria with No Investment

In absence of investment, the only decision for the firm is when to disclose. If the value function is increasing in beliefs, then the certification strategy is characterized by a certification threshold p_c. Let τ be the first time beliefs reach the certification threshold p_c. Direct computation yields the value function which is given by

$$V_L(p_t) = \int_t^\tau e^{-r(s-t)} p_s ds \tag{20}$$

$$V_H(p_t) = \int_t^\tau e^{-r(s-t)} p_s ds + e^{-(r+\lambda)(\tau-t)} \Big(V_H(p_0) - c \Big). \tag{21}$$

The certification threshold p_c is an equilibrium if and only if $V_H(p) \geq V_H(p_0) - c$ for all $p \geq p_c$ so the firm does not want to accelerate certification, and $V_H(p_c) \geq c$ so the firm's benefit of certification is higher than the cost.

Step 1: $V_H(p_c) \geq c$**.** Using (21) and $p_t = e^{-\lambda t} p_0 = e^{-\lambda t}$ we get

$$\begin{aligned} V_H(p_0) &= \frac{\int_0^\tau e^{-rs} p_s ds}{1 - e^{-(r+\lambda)\tau}} - \frac{e^{-(r+\lambda)\tau}}{1 - e^{-(r+\lambda)\tau}} c \\ &= \frac{1}{r+\lambda} - \frac{e^{-(r+\lambda)\tau}}{1 - e^{-(r+\lambda)\tau}} c, \end{aligned}$$

which is an increasing function τ and so a decreasing function of p_c (τ is decreasing in the threshold). Moreover, $V_H(p_0) \to -\infty$ as $\tau \to 0$; hence, there is a threshold p_c^+ such that $V_H(p_0) = c$. This means that p_c can be an equilibrium certification threshold only if $p_c \leq p_c^+$. Moreover, $p_c^+ > 0$ if and only if $c < \frac{1}{r+\lambda}$; otherwise, the unique equilibrium has no certification.

Step 2: $V_H(p) \geq V_H(p_0) - c$ **for all** $p \geq p_c$**.** A necessary condition for this to be the case is that $V_H'(p_c) \geq 0$; otherwise, there is ϵ such that $V_H(p_c + \epsilon) < V_H(p_0) - c$. If we differentiate (21) with respect to time we get

$$\begin{aligned} \frac{d}{dt} V_H(p_t) &= -p_t + r \int_t^\tau e^{-r(s-t)} p_s ds + (r+\lambda) e^{-(r+\lambda)(\tau - t)} \Big(V_H(p_0) - c \Big) \\ &= -p_t + r \int_t^\tau e^{-(r+\lambda)(s-t)} p_t ds + (r+\lambda) e^{-(r+\lambda)(\tau-t)} \left(\frac{1}{r+\lambda} - \frac{c}{1 - e^{-(r+\lambda)\tau}} \right) \\ &= e^{-(r+\lambda)(\tau-t)} \left(1 - \frac{r}{r+\lambda} p_t \right) - \frac{\lambda}{r+\lambda} p_t - \frac{c(r+\lambda) e^{-(r+\lambda)(\tau-t)}}{1 - e^{-(r+\lambda)\tau}}. \end{aligned}$$

Because p_t is decreasing in t we have that $V_H'(p_t) \geq 0$ if and only if $\frac{d}{dt} V_H(p_t) \leq 0$. This is true at time τ if and only if

$$\frac{d}{dt} V_H(p_t) \Big|_{t=\tau} = 1 - p_\tau - \frac{c(r+\lambda)}{1 - e^{-(r+\lambda)\tau}} \leq 0.$$

Using $p_\tau = p_c$ and $\tau = -\log(p_c)/\lambda$ we get the condition

$$1 - p_c - \frac{c(r+\lambda)}{1 - p_c^{\frac{r+\lambda}{\lambda}}} \leq 0 \tag{22}$$

The left hand side of equation (22) is decreasing in p_c. Hence, there is p_c^- such that (22) holds with equality if and only if $c \leq 1/(r+\lambda)$. Moreover, if this is the case, then condition (22) holds for any $p_c \geq p_c^-$. Hence, p_c^- is a lower bound for the certification threshold.

This is only a necessary condition; we still have to verify that $V_H(p) \geq V_H(p_0) - c$ for

$p > p_c$. Taking the second derivative of $V_H(p_t)$ we get

$$\begin{aligned}\frac{d^2}{dt^2}V_H(p_t) &= (r+\lambda)e^{-(r+\lambda)(\tau-t)}\left(1-\frac{r}{r+\lambda}p_t-\frac{c(r+\lambda)}{1-e^{-(r+\lambda)\tau}}\right)\\ &\quad-\left(e^{-(r+\lambda)(\tau-t)}\frac{r}{r+\lambda}+\frac{\lambda}{r+\lambda}\right)\dot{p}_t\\ &=(r+\lambda)\left(\frac{d}{dt}V_H(p_t)+\frac{\lambda}{r+\lambda}p_t\right)-\left(e^{-(r+\lambda)(\tau-t)}\frac{r}{r+\lambda}+\frac{\lambda}{r+\lambda}\right)\dot{p}_t\end{aligned}$$

Hence, we have that $\frac{d}{dt}V_H(p_t)=0$ implies $\frac{d^2}{dt^2}V_H(p_t)>0$. This means that if at time τ we have $\frac{d}{dt}V_H(p_t)\leq 0$ then it must be true that $\frac{d}{dt}V_H(p_t)\leq 0$ for all $t<\tau$. Thus, we have that

$$V_H(p_\tau)-V_H(p_t)=\int_t^\tau \frac{d}{dt}V_H(p_s)ds\leq 0,$$

so $V_H(p_t)\geq V_H(p_\tau)=V_H(p_0)-c$. The final step is to see in which situations the equilibrium has no investment.

Step 3: Investment Incentives

We can compute the incentives to invest using equations (20) and (21)

$$D(p_t)=e^{-(r+\lambda)(\tau-t)}\Big(V_H(p_0)-c\Big)=e^{-(r+\lambda)(\tau-t)}\left(\frac{1}{r+\lambda}-\frac{c}{1-e^{-(r+\lambda)\tau}}\right).$$

Hence, $D(p_t)<\frac{k}{\lambda}$ for all $t\leq\tau$ if and only if

$$\frac{1}{r+\lambda}-\frac{c}{1-e^{-(r+\lambda)\tau}}<\frac{k}{\lambda}.$$

This condition is true for any τ if and only if $\frac{1}{r+\lambda}-c<\frac{k}{\lambda}$. Otherwise, this is true if and only if

$$\tau<-\frac{1}{r+\lambda}\log\left(1-\frac{c}{\frac{1}{r+\lambda}-\frac{k}{\lambda}}\right),$$

which corresponds to the certification time τ consistent with the threshold p_c in the first part of Proposition 1.

Equilibria with Investment

We have already characterized the equilibria that have no investment. The final step is to look at those equilibria in which there is positive investment. The boundary conditions at p_c are given by

$$\begin{aligned} V_H(p_c) = V_H(0) &= V_H(1) - c \\ V_L(p_c) = V_L(0) &= \frac{\lambda a (V_H(1) - c) - ak}{r + \lambda a} \end{aligned} \tag{23}$$

Equation (23) can be rewritten

$$V_H(0) = \left(\frac{r}{\lambda a} + 1\right) V_L(0) + \frac{k}{\lambda},$$

hence

$$D(0) = \frac{r V_L(0)}{\lambda a} + \frac{k}{\lambda}.$$

On the other hand, $t \to D(p_t)$ is a continuous function so in equilibrium we must have that

$$D(p_c) = D(0) = \frac{k}{\lambda}.$$

Otherwise, the firm would invest when beliefs are just above p_c. We can thus conclude that

$$V_L(0) = V_L(p_c) = 0.$$

This in turn implies that

$$V_H(1) = \frac{k}{\lambda} + c.$$

Let $\tau = \inf\{p_t : p_\tau = p_c\}$. In equilibrium, $a(p_t) = 0$ implies that for $p_t > p_c$ we have

$$\tau = -\frac{\log p_c}{\lambda}.$$

The value function for the high type is given by

$$V_H(p_t) = \int_t^\tau e^{-r(s-t)} p_s + e^{-(r+\lambda)(\tau - t)} \big(V_H(1) - c\big) ds.$$

Using $p_s = e^{-\lambda(s-t)} p_t$ and $V_H(1) - c = k/\lambda$ we get

$$V_H(p_t) = \frac{p_t}{r+\lambda}\left[1 - \left(\frac{p_c}{p_t}\right)^{\frac{r+\lambda}{\lambda}}\right] + \frac{k}{\lambda}\left(\frac{p_c}{p_t}\right)^{\frac{r+\lambda}{\lambda}}.$$

Similarly,

$$V_L(p_t) = \frac{p_t}{r+\lambda}\left[1 - \left(\frac{p_c}{p_t}\right)^{\frac{r+\lambda}{\lambda}}\right].$$

Now, we can compute p_c using the condition $V_H(1) = c + k/\lambda$ which gives us

$$\left(\frac{1}{r+\lambda} - \frac{k}{\lambda}\right)\left[1 - p_c^{\frac{r+\lambda}{\lambda}}\right] = c,$$

so

$$p_c = \left[1 - \frac{c}{\frac{1}{r+\lambda} - \frac{k}{\lambda}}\right]^{\frac{\lambda}{r+\lambda}}.$$

Intuitively, p_c decreases in c and k. An equilibrium with certification and investment exists iff

$$\frac{1}{r+\lambda} - \frac{k}{\lambda} > c$$

Finally, no certification and no investment is an equilibrium if and only if

$$V_H^{nc}(0) > V_H^{nc}(1) - c,$$

which means that

$$c > \frac{1}{r+\lambda}.$$

□

B Proofs Section 4

Proof Lemma 2

Proof. For further reference, $x \equiv e^{-r\tau}$, $y \equiv e^{-r\tau_a}$, $\alpha \equiv (r+\lambda)/r$ and $q \equiv 1-\beta$ (and we simplify notation by not pointing out which θ_0 they correspond to since this is implied by the two cases we solve in sequence).

We start considering the case $\theta_0 = H$. The payoff of the high quality firm given an arbitrary tuple (x, y, q) is

$$\mathcal{U}_H(x,y,q) = \frac{1}{r+\lambda} + \frac{y-x}{r} - \frac{x^\alpha}{r+\lambda} + \frac{x^\alpha y^{1-\alpha} - y}{r+\lambda} - (y-x)\frac{k}{r} + x\left(1 - \left(\frac{x}{y}\right)^{\alpha-1} + x^{\alpha-1}\right)(\overline{U}_H - c)$$
$$+ x\left(\left(\frac{x}{y}\right)^{\alpha-1} - x^{\alpha-1}\right) q\overline{U}_L.$$

and the incentive compatibility constraint in terms of x, y, q is

$$y^\alpha = x^\alpha \left(\frac{\lambda(\overline{U}_H - c - q\overline{U}_L)}{k}\right).$$

For a fixed investment threshold τ_a, pinned down by y, we look for the optimal combination (x, q) that implements this y. Using the binding incentive compatibility constraint we get that for the fixed y

$$q'(x) = \frac{\alpha}{x}\frac{\overline{U}_H - c - q\overline{U}_L}{\overline{U}_L}.$$

The first derivative of $\mathcal{U}_H(x, y, q(x))$ with respect to x

$$\frac{\partial \mathcal{U}_H(x,y,q(x))}{\partial x} = -\left[\frac{1-k}{r} - (\overline{U}_H - c)\right] - \frac{1}{r+\lambda}\left[y - \alpha x^{\alpha-1}(y^{1-\alpha} - 1)\right].$$

so the second derivative is

$$\frac{\partial^2 \mathcal{U}_H(x,y,q(x))}{\partial x^2} = \frac{1}{r+\lambda}\alpha(\alpha-1)x^{\alpha-2}(y^{1-\alpha} - 1) > 0.$$

so $\mathcal{U}_H(x, y, q(x))$ is convex (by definition $y < 1$ and $\alpha > 1$) in x. This means that for an arbitrary y, the best pair (x, β) implementing y is an extreme point which, since $q(x)$ is increasing, means that we only need to consider $q = 0$ and $q = 1$.

The proof for the case $\theta_0 = L$ is analogous, with the minor difference that we can focus on $y = 0$ since we know it is optimal (as we argued in the text in a way independent of this lemma). The expected payoff of the firm is

$$\mathcal{U}_L(x,q) = \frac{(1-x)(1-k)}{r} + \frac{x^\alpha - 1}{r+\lambda} + x\left[1 - x^{\alpha-1}\right](\overline{U}_H - c) + x^\alpha q\overline{U}_L.$$

In the best equilibrium, the incentive compatibility constraint binds at $t = 0$ (since otherwise we could increase τ to save certification costs), so

$$1 = x^{\alpha}\left(\frac{\lambda(\overline{U}_H - c - q\overline{U}_L)}{k}\right),$$

or:

$$x^{\alpha} q \overline{U}_L = x^{\alpha}(\overline{U}_H - c) - \frac{k}{\lambda}.$$

Therefore, a q that satisfies the incentive compatibility constraint at $t = 0$ is increasing in x (intuitively, larger x means smaller τ so less time till certification, so the equilibrium can be more lenient without removing incentives for investment). Substituting q from this condition into $\mathcal{U}_L(x, q)$ we get:

$$\mathcal{U}_L(x, q(x)) = \frac{1-k}{r} - \frac{1}{r+\lambda} - \frac{k}{\lambda} - x\left(\frac{1-k}{r} - (\overline{U}_H - c)\right) + \frac{x^{\alpha}}{r+\lambda} \tag{24}$$

The second derivative is

$$\frac{d^2}{dx^2}\mathcal{U}(x, q(x)) = \frac{\alpha(\alpha-1)x^{\alpha-2}}{r+\lambda} > 0,$$

this means that the expected payoff is convex in x so the optimal q is again either zero or one. □

Proof Proposition 2

The proof of Proposition 2 follows the following steps:

- For $\theta_0 = H$
  - (i) First, we show that if $\overline{U}_H \geq 1/(r+\lambda) - c$ then the best equilibrium given $\beta_H = 1$ has full investment (Lemma 3) and $\tau_H^* = \tau_H^{FI}$.
  - (ii) Then we show that if c is small then the best equilibrium given $\beta_H = 1$ dominates the best equilibrium with $\beta_H = 0$ (Lemma 4),
  - (iii) and a solution to the equation $e^{-(r+\lambda)\tau_H^{FI}}(U_H^{FI}(\tau_H^{FI}) - c) = k/\lambda$ satisfying $U_H^{FI}(\tau_H^{FI}) - c \geq 1/(r+\lambda)$ exists (Lemma 5).
  - (iv) We conclude from the previous steps that for small c, $\tau_H^* = \tau_H^{FI}$ and $\overline{U}_H = U_H^{FI}(\tau_H^{FI})$.

- For $\theta_0 = L$

  (i) First, we show that a solution $\hat{U}_L^0$ to equation (19) exists (Lemma 6),

  (ii) and then show that $\beta_L = 0$ is optimal when c is small (Lemma 7).

- Finally, we show that a high quality firm has incentives to certify at time τ_θ^* (Lemma 8).

For reference, throughout the proofs we use notation $x \equiv e^{-r\tau}$ and $y \equiv e^{-r\tau_a}$, $q \equiv 1-\beta$ and $\alpha \equiv (r+\lambda)/r$, and we omit the reference to θ_0 since it is implied by the case described in each step.

Lemma 3. *Suppose $\overline{U}_H - c \geq 1/(r+\lambda)$ and $\beta_H^* = 1$. Then in the equilibrium that achieves $\overline{U}_H$, $\tau_H^* = \tau_H^{FI}$ and $\tau_a = 0$.*

Proof. Consider $\theta_0 = H$, $p_0 = 1$.

The incentive compatibility constraint that determines optimal investment policy can be written as:

$$\tau_a(\tau) = \inf\left\{t_a \in [0,\tau] : e^{-(r+\lambda)(\tau-t_a)}\big(\overline{U}_H - c\big) \geq \frac{k}{\lambda}\right\} = \max\left\{0, \tau - \frac{1}{r+\lambda}\log\left(\frac{\lambda\big(\overline{U}_H - c\big)}{k}\right)\right\}.$$

Let

$$\mathcal{U}_H(\tau,\tau_a(\tau),1) = \int_0^\tau e^{-rt}(p_t - \mathbf{1}_{t\geq\tau_a(\tau)}k)dt + e^{-r\tau}p_\tau(\overline{U}_H - c)$$

denote the equilibrium payoff for a given τ and for $\beta_H = 1$.

The best equilibrium for $\beta_H = 1$ implements full investment if

$$\tau_H^{FI} \in \arg\max_\tau \ \mathcal{U}_H(\tau,\tau_a(\tau),1)$$

Computing each individual term we get

$$\begin{aligned}\mathcal{U}_H(\tau,\tau_a,1) = {} & \frac{1}{r+\lambda} + \frac{e^{-r\tau_a} - e^{-r\tau}}{r} - \frac{e^{-(r+\lambda)\tau}}{r+\lambda} + e^{-r\tau_a}\frac{e^{-(r+\lambda)(\tau-\tau_a)} - 1}{r+\lambda} - \big(e^{-r\tau_a} - e^{-r\tau}\big)\frac{k}{r} \\ & + e^{-r\tau}\big(1 - e^{-\lambda(\tau-\tau_a)}(1 - e^{-\lambda\tau_a})\big)\big(\overline{U}_H - c\big)\end{aligned}$$

This expression is not convex in (τ,τ_a); for this reason, it is convenient to work with the transformed variables $x \equiv e^{-r\tau}$ and $y \equiv e^{-r\tau_a}$. Letting $\alpha \equiv (r+\lambda)/r$, we can write the payoff $\mathcal{U}_H(\tau, \tau_a, 1)$ as a function of the new variables (abusing notation for $\mathcal{U}$) as:

$$\mathcal{U}_H(x,y) = \frac{1}{r+\lambda} + \frac{y-x}{r} - \frac{x^\alpha}{r+\lambda} + \frac{x^\alpha y^{1-\alpha} - y}{r+\lambda} - (y-x)\frac{k}{r} + x\left(1 - \left(\frac{x}{y}\right)^{\alpha-1} + x^{\alpha-1}\right)(\overline{U}_H - c).$$

Let $x^* \equiv e^{-r\tau_H^{FI}}$. For $x \in [x^*, 1]$ we argued in the text that $\tau_a = 0$ and $x = x^*$ in this range is optimal. For any larger x, we do not get full investment, so $\tau_a > 0$ and the incentive compatibility constraint can be written in terms of x and y as

$$y = x \underbrace{\left(\frac{\lambda(\overline{U}_H - c)}{k}\right)^{\frac{1}{\alpha}}}_{M}.$$

Hence, for $x \geq x^*$, letting $\mathcal{U}_H(x) \equiv \mathcal{U}_H(x, y(x))$, where $y(x) = Mx$, we get:

$$\mathcal{U}_H(x) = \frac{1}{r+\lambda} + \frac{(M-1)(x-k)}{r} + \frac{(M^{1-\alpha} - M)x}{r+\lambda} + x(1 - M^{1-\alpha})(\overline{U}_H - c) + x^\alpha\left(\overline{U}_H - c - \frac{1}{r+\lambda}\right)$$

From here we get,

$$\mathcal{U}_H''(x) = \alpha(\alpha-1)x^{\alpha-2}\left(\overline{U}_H - c - \frac{1}{r+\lambda}\right)$$

So if $\overline{U}_H - c > \frac{1}{r+\lambda}$, then $\mathcal{U}_H(x)$ is convex. It implies that the maximum of $\mathcal{U}_H(x)$ is attained at an extreme point belonging to $\{0, x^*\}$. Finally, since

$$\mathcal{U}_H(0) = \frac{1}{r+\lambda}$$
$$\mathcal{U}_H(x^*) = (1-x^*)\frac{1-k}{r} + x^*\left(\overline{U}_H - c\right)$$

we get that, if $\overline{U}_H - c > \frac{1}{r+\lambda}$, then $x = x^* = e^{-r\tau_H^{FI}}$ is optimal. As a corollary, since $\overline{U}_H \geq U_H^{FI}(\tau_H^{FI})$, full investment is optimal for $\beta_H = 1$ whenever $U_H^{FI}(\tau_H^{FI}) - c > \frac{1}{r+\lambda}$. □

Lemma 4. *There is $\tilde{c}_1 > 0$ such that for any $c \leq \tilde{c}_1$ the payoff in the best equilibrium with $\beta_H = 1$ is higher than the highest payoff when $\beta_H = 0$.*

Proof. We can write the firm payoff as a function of (x, y, q) as (again abusing notation for $\mathcal{U}$):

$$\mathcal{U}_H(x,y,q) = \frac{1}{r+\lambda} + \frac{y-x}{r} - \frac{x^\alpha}{r+\lambda} + \frac{x^\alpha y^{1-\alpha} - y}{r+\lambda} - (y-x)\frac{k}{r} + x\left(1 - \left(\frac{x}{y}\right)^{\alpha-1} + x^{\alpha-1}\right)(\overline{U}_H - c)$$
$$+ x\left(\left(\frac{x}{y}\right)^{\alpha-1} - x^{\alpha-1}\right) q\overline{U}_L.$$

From the incentive compatibility constraint we have that

$$q\overline{U}_L = (\overline{U}_H - c) - \frac{k}{\lambda}\left(\frac{y}{x}\right)^\alpha.$$

which can be replaced in the firm's payoff to get

$$\mathcal{U}_H(x,y) = \frac{1}{r+\lambda} + \frac{y-x}{r} - \frac{x^\alpha}{r+\lambda} + \frac{x^\alpha y^{1-\alpha} - y}{r+\lambda} - (y-x)\frac{k}{r} + x(\overline{U}_H - c)$$
$$- x\left(\left(\frac{x}{y}\right)^{\alpha-1} - x^{\alpha-1}\right)\left(\frac{y}{x}\right)^\alpha \frac{k}{\lambda}.$$

Writing the incentive compatibility constraint for $q = 1$ as

$$y = x\left(\frac{\lambda(\overline{U}_H - c - \overline{U}_L)}{k}\right)^{\frac{1}{\alpha}} = xM$$

and substituting $y(x) = xM$ to $\mathcal{U}_H(x) \equiv \mathcal{U}_H(x, y(x))$ we get:

$$\mathcal{U}_H(x) = \frac{1}{r+\lambda} + \frac{(M-1)(1-k)x}{r} - \frac{x^\alpha}{r+\lambda} + x\frac{M^{1-\alpha} - M}{r+\lambda} + x(\overline{U}_H - c) - x\left(M^{1-\alpha} - x^{\alpha-1}\right)M^\alpha \frac{k}{\lambda}$$

Differentiating with respect to x we get that

$$\mathcal{U}'_H(x) = \frac{(M-1)(1-k)}{r} - \frac{\alpha x^{\alpha-1}}{r+\lambda} + \frac{M^{1-\alpha} - M}{r+\lambda} + (\overline{U}_H - c)$$
$$- \left(M^{1-\alpha} - \alpha x^{\alpha-1}\right)M^\alpha \frac{k}{\lambda}$$
$$\mathcal{U}''_H(x) = (\alpha-1)\alpha x^{\alpha-2}\left(\overline{U}_H - c - \overline{U}_L - \frac{1}{r+\lambda}\right)$$

We need to consider two cases: $\overline{U}_H - c - \overline{U}_L - \frac{1}{r+\lambda} > 0$ and $\overline{U}_H - c - \overline{U}_L - \frac{1}{r+\lambda} \leq 0$. In the first case, the payoff (given $q = 1$) is convex and so full investment is optimal (by the same reasoning as in the proof of Lemma 3). Moreover, with full investment it is optimal to set $q = 0$ as because this minimizes the certification cost. Let's assume then that that $\overline{U}_H - c - \overline{U}_L - \frac{1}{r+\lambda} \leq 0$. Let x_1 be the optimal x when $q = 1$. It must be the case that $x \in [0, M^{-1}]$ as any $x > M^{-1}$ implements the same investment as M^{-1} but at a higher certification cost. Under the assumption that $\overline{U}_H - c - \overline{U}_L - \frac{1}{r+\lambda} \leq 0$ the function $\mathcal{U}_H(x)$ is concave and so a necessary and sufficient condition for $x_1 = M^{-1}$ (so there is full investment, $y_1 = 1$) is that $\mathcal{U}'_H(M^{-1}) \geq 0$. We can compute:

$$\begin{aligned}
\mathcal{U}'_H(M^{-1}) &= \frac{(M-1)(1-k)}{r} - \frac{M}{r+\lambda} + (\overline{U}_H - c) - (\alpha - 1)\left(M^{1-\alpha}\frac{1}{r+\lambda} - M\frac{k}{\lambda}\right) \\
&= \frac{(M-1)(1-k)}{r} + (\overline{U}_H - c) - \frac{M}{r+\lambda} - \frac{M}{r}\left(\frac{k}{\overline{U}_H - c - \overline{U}_L}\frac{1}{r+\lambda} - k\right).
\end{aligned}$$

We want to show that $\mathcal{U}'(M^{-1}) \geq 0$ when $c \to 0$. With this objective in mind, we look for a lower bound for $\overline{U}_H - c - \overline{U}_L$. Note that

$$\begin{aligned}
\overline{U}_H - c &\geq U_H^{FI}(\tau_H^{FI}) - c \\
\overline{U}_L &\leq U_L^{FB} \equiv \frac{\lambda}{r+\lambda}\frac{1}{r} - \frac{k}{r},
\end{aligned}$$

where U_L^{FB} is the first best payoff. From here, we get that

$$\overline{U}_H - c - \overline{U}_L \geq U_H^{FI}(\tau_H^{FI}) - c + \frac{k}{r} - \frac{\lambda}{r+\lambda}\frac{1}{r}.$$

In the limit, when $c \to 0$ we have that $U_H^{FI}(\tau_H^{FI}) - c \to (1-k)/r = U_H^{FB}$. Accordingly, $\lim_{c\to 0}(\overline{U}_H - c - \overline{U}_L) \geq 1/(r+\lambda)$. Replacing in $\mathcal{U}'_H(M^{-1})$ we get that

$$\begin{aligned}
\lim_{c\to 0} \mathcal{U}'_H(M^{-1}) &\geq \frac{(M-1)(1-k)}{r} + (\overline{U}_H - c) - \frac{M}{r+\lambda} \\
&= (M-1)\left(\frac{1-k}{r} - (\overline{U}_H - c)\right) + M\left(\overline{U}_H - c - \frac{1}{r+\lambda}\right) > 0.
\end{aligned}$$

This means that for c small enough, $x = M^{-1}$ is optimal and so we have full investment and $q = 1 - \beta_H = 0$ being optimal. □

Lemma 5. *There is $\tilde{c}_2 > 0$ such that for any $c \leq \tilde{c}_2$ a solution to equation* (16) *satisfying $U_H^{FI}(\tau_H^{FI}) - c \geq 1/(r+\lambda)$ exists.*

Proof. First, we use the inequality $U_H^{FI}(\tau_H^{FI}) - c \geq 1/(r+\lambda)$ to find a lower bound for τ_H^{FI}.

Using equation (15) we get that $U_H^{FI}(\tau_H^{FI}) - c \geq 1/(r+\lambda)$ if and only if

$$\tau_H^{FI} \geq \underline{\tau} \equiv \frac{1}{r} \log\left(\frac{\lambda/(r+\lambda) - k}{\lambda/(r+\lambda) - k - rc}\right). \tag{25}$$

For future reference, remember that τ_H^{FI} solves

$$e^{-(r+\lambda)\tau}(U_H^{FI}(\tau) - c) = \frac{k}{\lambda}$$

Let

$$f(\tau) \equiv e^{-(r+\lambda)\tau}(U_H^{FI}(\tau) - c) - \frac{k}{\lambda} = e^{-(r+\lambda)\tau}\left(\frac{1-k}{r} - \frac{1}{1-e^{-r\tau}}c\right) - \frac{k}{\lambda},$$

so that by definition $f(\tau_H^{FI}) = 0$. An equilibrium with full investment satisfying the required properties exists if we can find $\tau \in [\underline{\tau}, \infty)$ such that $f(\tau) = 0$. The limit of $f(\tau)$ when τ goes to infinity is $\lim_{\tau\to\infty} f(\tau) = -k/\lambda < 0$, which means that it is enough to show that $f(\underline{\tau}) \geq 0$. If we evaluate $f(\tau)$ at the lower bound $\underline{\tau}$ we get

$$f(\underline{\tau}) = \left(\frac{\lambda/(r+\lambda) - k - rc}{\lambda/(r+\lambda) - k}\right)^{\frac{r+\lambda}{r}} \frac{1}{r+\lambda} - \frac{k}{\lambda}.$$

Given the parametric assumption $1/(r+\lambda) > k/\lambda$, the denominator in the last expression is positive, so the expression is decreasing in c and strictly positive for $c = 0$. Hence, $f(\underline{\tau}) > 0$ if $c \leq \tilde{c}_2$ where $\tilde{c}_2 > 0$ is chosen such $f(\underline{\tau}) = 0$. □

Lemma 6. *Suppose that $\overline{U}_H - c \geq 1/(r+\lambda)$ then there is $\hat{U}_L^0 \in (0, \overline{U}_H - c - k/\lambda)$ such that*

$$\hat{U}_L^0 = \int_0^{\tau_L^0} e^{-rt}(p_t^L - k)dt + e^{-r\tau_L^0}\left(p_{\tau_L^0}^L(\overline{U}_H - c) + (1 - p_{\tau_L^0}^L)\hat{U}_L^0\right)$$
$$\tau_L^0 = \frac{1}{r+\lambda} \log\left(\frac{\lambda(\overline{U}_H - c - \hat{U}_L^0)}{k}\right).$$

Proof. Let's define the function

$$G(u) = \int_0^{\tau(u)} e^{-rt}(p_t^L - k)dt + e^{-r\tau(u)}\left(p_{\tau(u)}^L(\overline{U}_H - c) + (1 - p_{\tau(u)}^L)u\right) - u$$

where

$$\tau(u) = \frac{1}{r+\lambda} \log\left(\frac{\lambda(\overline{U}_H - c - u)}{k}\right)$$

We need to show that a solution $G(u) = 0$ exists on the open interval $(0, \overline{U}_H - c - k/\lambda)$ (the restriction that $\overline{U}_L$ is strictly lower than $\overline{U}_H - c - k/\lambda$ is required to guarantee that $\tau > 0$). Noting that $G(\overline{U}_H - c - k/\lambda) = 0$ and $G(0) = \hat{U}_L^1 > 0$ we conclude that it is enough to show that $G(\overline{U}_H - c - k/\lambda - \epsilon) < 0$ for some small $\epsilon > 0$. Because $G(u)$ is continuous, it is sufficient to show that $G'(\overline{U}_H - c - k/\lambda) > 0$. For convenience, we use the change of variable $x(u) \equiv e^{-r\tau(u)}$ and write

$$G(u) = \frac{(1-x)(1-k)}{r} + \frac{x^\alpha - 1}{r+\lambda} + x\left[1 - x^{\alpha-1}\right](\overline{U}_H - c) + x^\alpha u - u$$

where as usual $\alpha \equiv (r+\lambda)/r$. Using the incentive compatibility constraint we can verify that

$$x'(u) = \frac{x(u)}{\alpha(\overline{U}_H - c - u)}.$$

Differentiating $G(u)$ we get

$$G'(u) = x'(u)\left[\overline{U}_H - c - \frac{(1-k)}{r} + \frac{x^{\alpha-1}}{r}\right] + 2x^\alpha - 1$$

Evaluating at $\hat{u} = U_H - c - \frac{k}{\lambda}$ we get

$$G'(\hat{u}) = x'(\hat{u})\left[\overline{U}_H - c + \frac{k}{r}\right] + 1 > 0$$

As $G(\hat{u}) = 0$ and $G(0) = \hat{U}_L^1 > 0$ there is $\hat{U}_L^0 \in (0, \hat{u})$ such that $G(\hat{U}_L^0) = 0$. □

Lemma 7. *There is $\tilde{c}_3 > 0$ such that $\beta_L = 0$ is optimal for all $c \leq \tilde{c}_3$.*

Proof. Fix $\theta_0 = L$.

We want to show that when $c \to 0$, $q = 1 - \beta = 1$ is optimal. Consider the firm's payoff after replacing the binding incentive compatibility constraint (recall that in case $\theta_0 = L$ in the best equilibrium $\tau_a = 0$, so this expression uses $y = 1$.)

$$\mathcal{U}_L(x) \equiv \mathcal{U}_L(x, q(x)) = \frac{1-k}{r} - \frac{1}{r+\lambda} - \frac{k}{\lambda} - x\left(\frac{1-k}{r} - (\overline{U}_H - c)\right) + \frac{x^\alpha}{r+\lambda}.$$

Note it is convex and the derivative is

$$\mathcal{U}_L'(x) = -\left(\frac{1-k}{r} - (\overline{U}_H - c)\right) + \frac{\alpha x^{\alpha-1}}{r+\lambda}$$

Let $x^0 = x(q=0)$ and $x^1 = x(q=1)$ and recall that $x^1 > x^0$. If we replace x^0 and α we get

$$\mathcal{U}_L'(x^0) = -\left(\frac{1-k}{r} - (\overline{U}_H - c)\right) + \frac{1}{r}\left[\frac{k}{\lambda}(\overline{U}_H - c)\right]^{\frac{\alpha-1}{\alpha}}.$$

It is straightforward to show that $U_H^{FI}(\tau_H^{FI})$ converges to the first best payoff $\frac{1-k}{r}$ as c goes to zero because the frequency of certification remains bounded:

$$\lim_{c\to 0}\ \tau_H^{FI} = \frac{1}{r+\lambda}\log\left(\frac{1-k}{r}\frac{\lambda}{k}\right) > 0.$$

Therefore $\lim_{c\to 0}(\overline{U}_H - c - (1-k)/r) = 0$ which means that $\lim_{c\to 0}\mathcal{U}_L'(x^0) > 0$. The optimality of x^1 follows from the convexity of $\mathcal{U}_L(x)$. □

Lemma 8. *It is never optimal for a high quality firm to delay certification at time* τ_θ^*

Proof. In the case of τ_H^* it is straightforward that the firm would not deviate as the deviation payoff is zero (the reputation drops to $p=0$ and even if the firm certifies later, it has to pay c and receive continuation payoff $\underline{U}_H = c$ for a net payoff 0). The same reasoning applies if τ_L^* and $\beta = 1$, i.e. if the equilibrium is harsh. The case of τ_L^* is a bit different when the equilibrium is lenient, $\beta = 0$ because the high quality firm can then deviate to certification at some other on-path time, for example $2\tau_L^*$ (the previous reasoning applies if the firm deviates to off-path time). It is sufficient to consider a single-step deviation in which the firm that does not certify at time τ_L^* certifies for sure at time $2\tau_L^*$. The payoff of such a deviation is

$$\tilde{U}_H = \int_0^{\tau_L^*} e^{-rt}(p_t^L - k)dt + e^{-r\tau}(\overline{U}_H - c)$$

Adding and subtracting $(1-p^L_{\tau_L^*})\overline{U}_L$ we can write

$$\begin{aligned}
\tilde{U}_H &= \int_0^{\tau_L^*} e^{-rt}(p_t^L - k)dt + e^{-r\tau_L^*}\left(p^L_{\tau_L^*}(\overline{U}_H - c)) + ((1-p^L_{\tau_L^*})\overline{U}_L)\right) + e^{-r\tau_L^*}(1-p^L_{\tau_L^*})(\overline{U}_H - c - \overline{U}_L) \\
&= \overline{U}_L + e^{-r\tau_L^*}(1-p^L_{\tau_L^*})(\overline{U}_H - c - \overline{U}_L) \\
&= \left(1 - e^{-r\tau_L^*}(1-p^L_{\tau_L^*})\right)\overline{U}_L + e^{-r\tau_L^*}(1-p^L_{\tau_L^*})(\overline{U}_H - c) \\
&< \overline{U}_H - c,
\end{aligned}$$

which means that a high quality firm never has incentives to delay certification at $t = \tau_L^*$. □

B.1 Comparative Statics

In this section we show that $q = 1 - \beta_L$ is non-increasing in c. Replacing the binding IC constraint, we get that the payoff of a low quality firm given (x, q) (recall $x = e^{-r\tau}$) is

$$\mathcal{U}_L(x, q(x)) = \frac{1-k}{r} - \frac{1}{r+\lambda} - \frac{k}{\lambda} - x\left(\frac{1-k}{r} - (\overline{U}_H - c)\right) + \frac{x^\alpha}{r+\lambda}.$$

We show that q is non-increasing by using monotone comparative static. Let $\mathcal{U}_L(x, q(x), c)$ be the payoff of the low quality firm given by equation (24) as a function of c. The cross derivative with respect to c and x is

$$\frac{\partial^2}{\partial x \partial c}\mathcal{U}_L(x, q(x), c) = \frac{\partial}{\partial c}(\overline{U}_H(c) - c) = \overline{U}'_H(c) - 1 < 0.$$

Thus, $\mathcal{U}_L(x, q(x), c)$ satisfies the single crossing property. Using monotone comparative statics we conclude that x is non-increasing in c. Combining the fact that $x = e^{-r\tau}$ and that τ is higher when $q = 0$ we verify that τ is non-decreasing in c. But then the incentive compatibility constraint immediately implies that q is non-increasing in c.